COMENIUS UNIVERSITY
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

Fractal dimension of 23 selected images, 18 with a mathematical theme

Bachelor Thesis

Branch of study:	MATHEMATICS
Supervisor:	doc. RNDr. Andrej Ferko, PhD.
Consultant:	RNDr. Róbert Bohdal, PhD.

Bratislava, 2023

Alexandra Dyalee

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THESIS ASSIGNMENT

Name and Surname:	Alexandra Dyalee
Study programme:	Mathematics (Single degree study, bachelor I. deg., full time form)
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Type of Thesis:	Bachelor´s thesis
Language of Thesis:	English
Secondary language:	Slovak

Title: Fractal dimension of 23 selected images, 18 with a mathematical theme

Annotation: Escher, Fomenko and many other painters depict mathematical subjects. In the project, we compare selected works using the box counting method. The hypothesis is that the approximate fractal dimension will make it possible to characterize the approaches of different authors.

Aim: 1. Study and overview of the issue, CRUSE scanner, selection of data and processing options.
2. Specification of the software work.
3. Experiments and evaluation, possibly publishing at ŠVK 2023 conference.

Literature: BARNSLEY, M. 1988. Fractals Everywhere. Boston: Academic Press.
MANDELBROT, B. 2014. Fraktalista. Praha: Argo.
Virtual Math Museum [online] http://virtualmathmuseum.org/mathart/ArtGalleryAnatoly/Anatolyindex.html
Andrej FERKO, Martin SAMUELČÍK, Veronika ŠPRLÁKOVÁ. 2015. GEOMETRIA FRAKTÁLOV. FMFI UK Bratislava. [online] http://www.sccg.sk/~ferko/GeometriaFraktalovFinalCD.pdf
RUŽICKÝ, E. – FERKO, A. 1995. Počítačová grafika a spracovanie obrazu. Bratislava: SAPIENTIA.
[Online] http://www.sccg.sk/~ferko/PGASO2012-bookmarks.pdf.

Keywords: Fractal geometry, box counting dimension, CRUSE scanning

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ZADANIE ZÁVEREČNEJ PRÁCE

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Jazyk záverečnej práce: anglický
Sekundárny jazyk: slovenský

Názov: Fractal dimension of 23 selected images, 18 with a mathematical theme
Fraktálna dimenzia 23 vybraných obrazov, 18 s matematickou tematikou

Anotácia: Escher, Fomenko i mnohí ďalší maliari zobrazujú matematické námety. V projekte porovnávame vybrané diela pomocou box counting method. Hypotézou je, že približná fraktálna dimenzia umožní charakterizovať prístupy rôznych autorov.

Cieľ: 1. Štúdium a prehľad problematiky, skener CRUSE, výber dát a možností spracovania.
2. Špecifikácia softverového diela.
3. Experimenty a vyhodnotenie, prípadne publikovanie na ŠVK.

Literatúra: BARNSLEY, M. 1988. Fractals Everywhere. Boston: Academic Press.
MANDELBROT, B. 2014. Fraktalista. Praha: Argo.
Virtual Math Museum [online] http://virtualmathmuseum.org/mathart/ArtGalleryAnatoly/Anatolyindex.html
Andrej FERKO, Martin SAMUELČÍK, Veronika ŠPRLÁKOVÁ. 2015. GEOMETRIA FRAKTÁLOV. FMFI UK Bratislava. [online] http://www.sccg.sk/~ferko/GeometriaFraktalovFinalCD.pdf
RUŽICKÝ, E. – FERKO, A. 1995. Počítačová grafika a spracovanie obrazu. Bratislava: SAPIENTIA.
[Online] http://www.sccg.sk/~ferko/PGASO2012-bookmarks.pdf.

Kľúčové slová: Fractal geometry, box counting dimension, CRUSE scanning

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Dátum zadania: 25.09.2022

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študent vedúci práce

ABSTRACT

Conception of fractal geometry has been embellishing our knowing as such since the novelty era of Mandelbrot, in 1980s, yet the fractal world had been here with us since ever. As viewers of mother nature, we have an innate ability to percept some effigy of a great many of things created by it, perhaps granting us with survival in the great big mosaic as we are tiny particles of this all. But there is a very fine line between the way we do see and might tend to simplify the nature, and the way it really is. Consider a simple snowflake – if we try to divide its shape to parts, we end up with a nearly identical copy of the whole snowflake, just with reduced size! Further, one might end up in raptures about the way we can keep getting closer and closer, but never really get enough close to surely claim we cannot get any closer – saving the possibility that this would be exactly the case underlying the possibility of existence…

Either way, we have one more interesting issue to be taken into consideration ruminating of nature, not really having been investigated enough to be up to now understood precisely. Within living memory, there have existed a group of people, for each era specific, called artists. Capturing the prominently deep sentiment, they have been adapting exactly toward the form of unity with nature, speaking nothing of the cases there was no need to adapt. To be particular, let's cite Kant on genius: "Genius is the innate mental aptitude (ingenium) through which nature gives the rule to art." Finally, the question to arise is following: could we say with accuracy what is that very thing art has to do with nature, and what is the (exiguous) thing nature has to do with art?

Maybe, with the term of understanding the core of nature with fractal dimension, we could advance toward truth a step closer – in this ambition, we decided to examine the interconnection between these two. Mathematical tools used in such research are all about the fractal geometry of nature; fractal dimension on natural patterns and box counting dimension providing us the approximate fractal dimension of fine art pieces.

ABSTRAKT

Koncepcia fraktálnej geometrie obohacuje naše poznanie ako také – ako celok, ako hodnotu dosiahnutú ľudskou inteligenciou – už od 80-tých rokov, kedy sa ňou nechala inšpirovať priekopnícka a bezpochyby výnimočná osobnosť Mandelbrota – cez to však v rozdielnom pohľade na vec musíme konštatovať, že fraktálny svet tu s nami existuje odjakživa. My vo všeobecnosti na prírodné princípy máme možnosti reagovať – ako publikum veľkolepej matky prírody disponujeme vrodenou schopnosťou vnímať časť z toho veľkého množstva, ktoré ona sama vytvorila – zrejme už aj preto, aby sme dokázali prežiť vo veľkej mozaike niečoho, čoho malinkou súčasťou máme vlastne česť byť. Avšak, medzi týmito dvomi – medzi tým, ako my prírodné princípy vnímame a uchopujeme, a pre lepšie porozumenie si ich môžeme mať tendenciu zjednodušovať, a tým, ako v skutočnosti sú konštruované, je význačný a spektakulárny rozdiel.

Pozrime sa len na jednoduchú snehovú vločku – ak sa jej tvar pokúsime rozdeliť na menšie časti, dostaneme takmer identické podobizne pôvodného tvaru, jediný rozdiel bude vo veľkosti. Nahliadnuc sem s odstupom, malo by nás priam ohromiť, ako je možným blížiť sa stále bližšie a bližšie, no nikdy sa nepriblížiť dostatočne blízko na to, aby sme mohli tvrdiť, že už nemožno dôjsť bližšie – vyminúc fakt, že by práve toto mohlo byť princípom stojacim za (našou) samotnou existenciou...

Tak, či onak, uvažujúc prírodu máme ešte jeden problém, ktorý dodnes naším poznaním nie je naplno uchopený. Od najútlejšieho veku ľudstva existovala skupinka ľudí, špecifická pre každú dobu a hnutie, nazývaná umelcami. Zachytávajúc hĺbku v mimoriadnosti odhodlania, s prírodou vytvárali jednotu, nehovoriac o prípadoch, kedy tou jednotou boli v takom kontexte, v akom je axióma základným stavebným kameňom akejkoľvek matematickej teórie. Aby sme hovorili presne, budeme citovať Kanta vo výklade geniality: "Génius nie je ničím iným, než je vrodenou mentálnou schopnosťou, skrz ktorú príroda dáva pravidlo umeniu."

Konečne, otázka pre nás je nasledovná: dokázali by sme v presnosti a určitosti povedať, čo je práve tou jednou vecou, ktorá spája prírodu s umením, a čo je tou vecou, ktorá naopak spája umenie s prírodou?

Možnože, ak by sa nám podarilo porozumieť srdcu stavby prírodnej štruktúry prostredníctvom neoblomného aparátu matematiky a fraktálnych dimenzií, bolo by to práve tým krokom, ktorý nás posunie o jeden dôležitý krok vpred ku pravde. Práve preto sme sa rozhodli pre výskum, ktorým budeme skúmať ich formálne (matematické) prepojenie. Aparát, ktorý pre tento cieľ využijeme, bude celý spočívať vo fraktálnej geometrii; základom je fraktálna dimenzia prírodných vzorov, ktorú však posunieme vyššie do tzv. box counting dimenzie, ktorá nám ponúkne približné fraktálne dimenzie výtvarných diel.

Contents

1 INTRODUCTION

Our thesis came into being in my personal opinion by an exotic motive, which is a mathematical inquiry for a possible formal interconnection of the worlds of nature and fine arts. For the sake of our aim to possibly make this question clearer, we make use of the phenomena of fractal geometry, that is up to now considered to be the most accurate tool for describing natural patterns.

We proceed by collecting artworks from multiple authors and of varying qualitative features, and design three experiments based on fractal dimensions of binary versions of respective artworks. Having done this, we continue to analyse respective experiments and quest after the evidence of fractal features of artworks and their qualitative entries.

Questions to be answered by our analysis are stated in a following way:

* *Question 1:* What is an "usual" fractal dimension of an artwork?
* *Question 2:* Does the expected fractal dimension of an artwork depend upon the author?
* *Question 3:* Is there a difference between fractal dimensions of differently elaborated artworks?
* *Question 4:* What is the particular number of fractal dimension revealing about the artwork?

Allured by curiosity and avidity for finding out of the answers, we come through the research work step by step, adhering to a structure we established beforehand. It comprises of three main chapters linked with three main streams of thought – first of them is *Chapter 1: FEW IDEAS TO ADORE – FURTHERMORE PICK UP ON*. Here, we describe: the very idea behind and mathematical basis of fractal geometry, selected artworks and motives behind their selection, and methods we use to digitalize them. Having absorbed this theoretical background, we move to *Chapter 2: WORK IN (PROGRESS) PROCESS*, where we describe the particular software features needed for fractal analysis of digitalized artworks and design our experiments. Finally, in *Chapter 3: DID WE SUCCEED?*, we interpret the experiments with respect to used methods and that way we try to provide at least partial answers on established questions.

2 NOTATION AND METADATA

2.1 Notation

Subsection 3.3.1:

ϵ - yardstick length

$L(\epsilon)$ - coastline length

D - dimension

Subsection 3.1.2:

$D_{capacity}$ - capacity dimension

D_T - topological dimension

$\mathcal{H}(\mathbb{X})$ - family of metric outer measures on $\mathbb{X}$

$(\mathbb{X}, d)$ - metric space $\mathbb{X}$ with metric d

$\mathcal{N}(A, \epsilon)$ - smallest number of closed balls of radius $\epsilon > 0$ needed to cover A

$\mathcal{N}_n(A)$ - number of boxes of side length $\frac{1}{2^n}$ intersecting A

Subsection 3.4 (meta-language):

$\beta_{\mathbb{X}}$ - unity consisting of a set of k amounts of energy

$\mathbb{X}$ - bunch label

2.2 Metadata

Subsection 3.3: Nine Artworks of Anatoly Fomenko
TITLE (SERIES, No. in series. Year of completion. *Mathematical theme portrayed.* Medium, parameters.) Parameters of digital version.

Subsection 3.4: Nine Artworks of Alexandra Dyalee
TITLE IN LATIN (SERIES, No. in series. Year of completion. TITLE IN ENGLISH, type. *Mathematical phenomenon metaphorized.* Medium, parameters.) Parameters of digital version. AWARD/FEATURE (if applicable).

Subsection 3.5: Miscellaneous from the Faculty
TITLE (Author, year of completion. Medium, parameters.) Source. Parameters of digital version.

3 FEW IDEAS TO ADORE – FURTHERMORE PICK UP ON

3.1 Fractal Geometry

3.1.1 How Long is the Coast of Great Britain?

One of the stupendously compelling investigations in the field of Geometry of Nature began in the momentum as humankind was first asked to decide on the length of a property diffusing with geography – the coast of Great Britain.

As scheduled by B. Mandelbrot [1], there were multiple alternative methods of measurement considered, using a yardstick length ϵ, but all misbehaving in the fact that putting it into best possible precision, the investigated length approaches to infinity. The reason got clear as Richardson [2], (1961) basing upon empirical research, approximated the length of the coastline in dependence of ϵ to be $L(\epsilon) \sim F_{\epsilon}^{2-D}$, where F_{ϵ}^{-D} is a rough number of intervals of length ϵ needed for the respective approximation. However, he didn't assign the constant D any particular significance. Such state of the problem has been further grasped by Mandelbrot, who made a mare's nest around the field with one whack – he did interpret the exponent D as a *dimension*.

On the ground of this, the dimension D regarding the case of the coast of Great Britain has been shown to exceed 1, whereby the dimension equal to 1 is the intuitive dimension of curves (and also their topological dimension). Such phenomenon has been denominated by Mandelbrot as a *fractal curve* and, as any coastline is a pattern possible to be modeled by fractal curves, we state **coastlines are fractal patterns**.

3.1.2 Definition of a Fractal

Informally, Mandelbrot's fractal geometry [6] is a (new) geometry of nature fully able to describe many of the irregular and fragmented patterns around us, which is a task out of the capacity of up to Mandelbrot's time, classical, euclidean geometry. Exactly for this reason, geometry has had a tendency to be described as "cold" or "dry" for a long time, as it could describe only the ideal shapes, as are circles or cubes, which definitely *are* amazing, but can cover only a small part of all of the geometry of nature. A perfect demonstration of this is an above coast of Great Britain.

Formally (and accurately), there are accepted following definitions.

Definition 3.1. *Fractals.*
Fractals are objects whose capacity dimension $D_{capacity}$ *is different from their Lebesgue covering dimension (topological dimension)* D_T.

The main difference between these two approaches [7] toward defining and understanding the meaning of dimension is the fact that topological dimension D_T is always an integer, while capacity dimension $D_{capacity}$ (also known as fractal dimension, Hausdorff dimension, and Hausdorff-Besicovitch dimension) does not need to be an integer. We conclude furthermore that for non-fractal objects (as are for example sets in Euclidean space) holds $D_{capacity} = D_T$.

Definition 3.2. *Hausdorff-Besicovitch dimension (fractal dimension).*
The capacity dimension of the set $A \in \mathcal{H}(\mathbb{X})$, where $(\mathbb{X}, d)$ is a metric space, is a real number $D_{capacity}$ such that

$$D_{capacity} = \lim_{\epsilon \to 0} \left\{ \frac{\ln \mathcal{N}(A, \epsilon)}{\ln \left(\frac{1}{\epsilon}\right)} \right\}$$

(if the limit exists), where $\mathcal{N}(A, \epsilon)$ is the smallest number of closed balls of radius $\epsilon > 0$ needed to cover A.

To briefly introduce the concept of topological dimension, we will only use the example of a space $\mathbb{R}^n$ with topological dimension n, that will be fully sufficient for our work in the affairs of having an intuitive conception of it.

3.1.3 Box-Counting Method

As stated in [9], we will move on to outline the Box Counting Theorem, which could be with impunity labeled as the most crucial (technical) point in our work. For the sake of our appraise, we will also prove the theorem.

Theorem 3.1. *The Box Counting Theorem. Let $A \in \mathcal{H}(\mathbb{R}^m)$, where the Euclidean metric is used. Cover $\mathbb{R}^m$ by closed square boxes of side length $(\frac{1}{2^n})$, as exemplified in Figure 1 below for $n = 2$ and $m = 2$. Let $\mathcal{N}_n(A)$ denote the number of boxes of side*

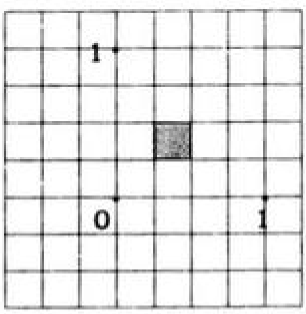

Figure 1: Cover by square boxes of size 2^{-n}

length $(\frac{1}{2^n})$ which intersect the attractor. If

$$D = \lim_{n \to \infty} \left\{ \frac{\ln \mathcal{N}_n(A)}{\ln 2^n} \right\}$$

then A has fractal dimension D.

Proof. We observe that for $m = 1, 2, 3, \dots$ is

$$2^{-m}\mathcal{N}_{n-1}(A, 2^{-n}) \leq \mathcal{N}(A, 2^{-n}) \leq \mathcal{N}_{k(n)}(A, 2^{-n})$$

for all $n = 1, 2, 3, \dots$, such that $k(n)$ is the smallest integer k satisfying $k \leq n - 1 + \frac{1}{2}\log_2 m$. The first inequality holds because a ball of radius 2^{-n} can intersect at most 2^m "on grid" boxes of side 2^{1-n}. The second follows from the fact that a box of side s can fit inside a ball of radius r provided $r^2 \geq (\frac{s}{2})^2 + (\frac{s}{2})^2 + \cdots + (\frac{s}{2})^2 = m(\frac{s}{2})^2$ by Pythagorean's Theorem. Now

$$\lim_{n\to\infty}\left\{\frac{\ln(\mathcal{N}_{k(n)})}{\ln(2^n)}\right\} = \lim_{n\to\infty}\left\{\frac{\ln(2^{k(n)})}{\ln(2^n)}\frac{\ln(\mathcal{N}_{k(n)})}{\ln(2^{k(n)})}\right\} = D,$$

since $\frac{k(n)}{n} \to 1$. Since also

$$\lim_{n\to\infty}\left\{\frac{\ln(2^{-m}\mathcal{N}_{n-1})}{\ln(2^n)}\right\} = \lim_{n\to\infty}\left\{\frac{\ln(\mathcal{N}_{n-1})}{\ln(2^{n-1})}\right\} = D,$$

setting $r = \frac{1}{2}$ into Theorem 1.1* of [9], we see

$$\lim_{n\to\infty}\left\{\frac{\ln\mathcal{N}(A, C.2^{-n})}{\ln\frac{2^n}{C}}\right\} = D$$

for any real $C > 0$, so for $C = 1$ will hold our theorem under consideration.

*This theorem says that setting $\epsilon_n = Cr^n$ for any real numbers $0 < r < 1$ and $C > 0$, $A \in \mathcal{H}(\mathbb{X})$ for a metric space $(\mathbb{X}, d)$, the fractal dimension of A is

$$D = \lim_{n\to\infty}\left\{\frac{\ln\mathcal{N}(A, \epsilon_n)}{\ln\left(\frac{1}{\epsilon_n}\right)}\right\}$$

□

Up to this point, we have been discussing predominately the undoubtedly stunning mathematical foundation of the very possibility of box counting measurements being fractal. Moving a bit forward, taking the help of Peitgen, et al [11], we will demonstrate how is the box counting method applicable to computational use.

We will be having to compute a fractal dimension of shapes very irregular, having one and only property, and so – *they don't have properties we could generalise.* Example of this can be seen in the below Figure 2.

Let's go ahead with the algorithm. We put the structure A onto a grid with mesh size s and count the number $\mathcal{N}_n(A)$ of grid boxes which contain some of the structure. Now, instead of setting $n \to \infty$, we change s to progressively smaller sizes, and count

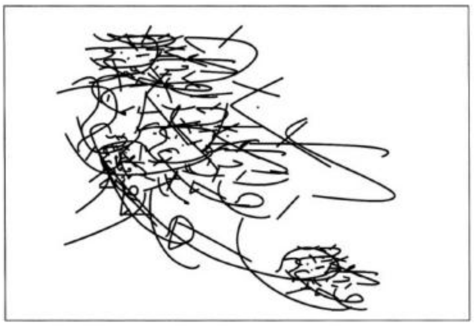

Figure 2: A wild fractal [Peitgen]

the corresponding numbers of grid boxes. Therefore, instead of $\mathcal{N}_n(A)$, we will write $\mathcal{N}(s)$ for our sake, taking in mind that the number of boxes n covering A do immediately imply the grid mesh size s.

Having determined enough of counts $\mathcal{N}(s)$ for corresponding s, we move toward a log/log diagram, in which we plot the logarithms $\log \mathcal{N}(s)$ versus $\log \frac{1}{s}$. Then, we *try to fit a straight line* to the plotted points of the diagram and measure its slope, which will be our (box-counting) fractal dimension D_b. Example of this is shown in the Figure 3 below.

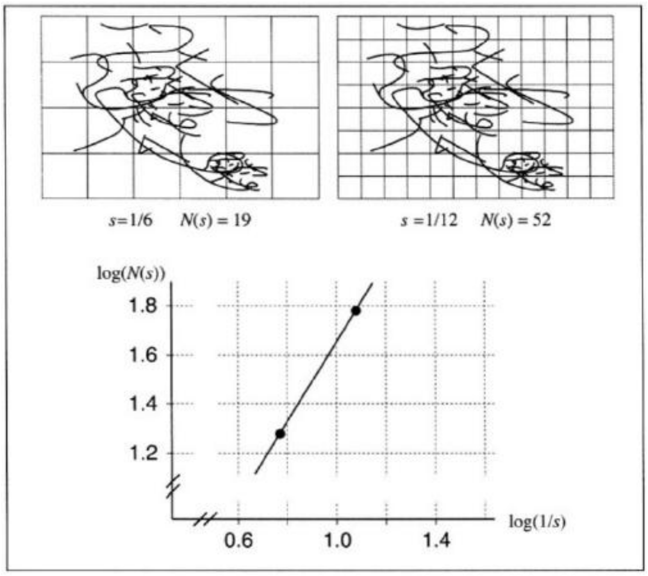

Figure 3: The wild structure box-counted using two grids

Practically, it is often very convenient to consider such a sequence of grids where the mesh is reduced to half from one grid to the next. Reason is such; using this approach, each box forming a grid is subdivided into four boxes, each of half the size s in the next grid. Onwardly, we arrive at sequence of counts $\mathcal{N}(2^{-k})$ for $k = 0, 1, 2, \ldots$. Arising out of this, there has been adopted the convention to set $s = 2^0 = 1$ for the coarsest grid.

Furthermore, slope of the line from one data to the next in the corresponding log/log diagram is

$$\frac{\log\left(\mathcal{N}(2^{-(k+1)})\right) - \log\mathcal{N}(2^{-k})}{\log 2^{k+1} - \log 2^{k}} = \log_2 \frac{\mathcal{N}(2^{-(k+1)})}{\mathcal{N}(2^{-k})}$$

discharging that the logarithm on the right side works for base 2, while the term on the left holds for any base.

There is one more important thing to underdraw about such approach toward calculation of the slopes; **this slope is an estimate for the box-counting dimension of the fractal.** In other words, if the number of boxes counted increases by a factor of 2^D when the box size is halved, then the fractal dimension is equal to D.

3.2 Scanning Fine Arts with CRUSE

No doubt every (not only) fine-art artist does feel a huge sense of personal care about the way any of their masterwork will be digitized, if it is about to be digitalized or reproduced any further – this could be at very ease comparable to a serious concept of parenthood.

For the sake of acquitting this, there has been developed a state-of-the-art-technology for scanning fine art pieces of broad spectrum of mediums. Presently, *that* best system worldwide is a **Cruse Synchrontable Scanner.**

Cruse Synchrontable Scanner is a scanning technology preserving as the original textures, as the maximum color accuracy with the scanned original, which is all based on its extensive lighting options. Particular technical features of Cruse Scanner, which are crucial for acquiring the best needed quality of artwork samples used for our research work do include [3]

· a 15K CMOS trilinear sensor with 5.6 μm pixel size,

· resolutions up to 2000 pixels per inch possible, and

· different light modes: left + right, texture effect, light angle device, rear light.

The process of scanning our artworks (see illustrated on below PRINT-SCREEN (Figure 4)) consists of the following steps:

1. We turn on the **processing computer**. This one is characteristic most importantly for a superior display screen and newer both hardware and software, e.g. GIMP. Except for the process of scanning, it does access a drive of the *controller computer.*

2. We turn on the **controller computer**. This device is responsible for the execution of both mechanical and optical adjustment of the scanner. These include for us the table position, head height, gain and integration time.

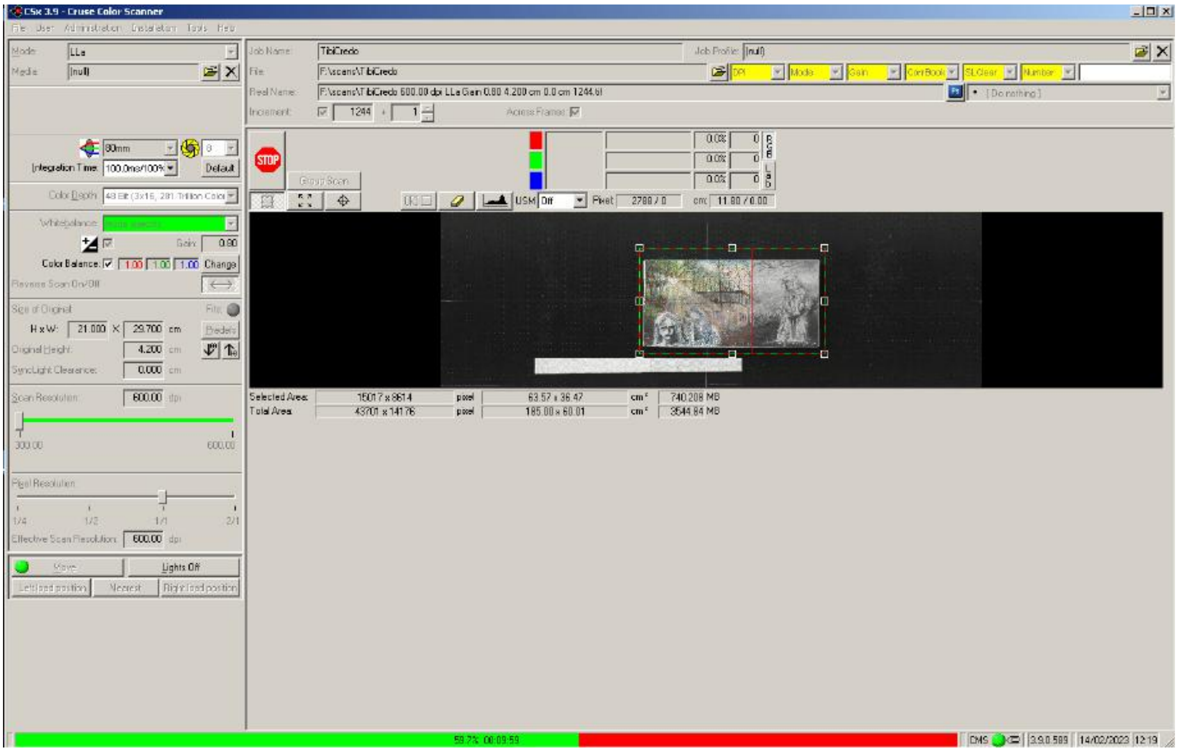

Figure 4: Process of scanning with CRUSE

On the bus, there is a custom-made card linking it with a *media center*.

3. We turn on the **media center** by spinning the power switch from value 0 to value 1, that triggers a switch-in of the current into scanner.

4. We turn on the **scanner** with a green press button, which is located at the bottom, underneath the table. If the scanner has been last switched off by an emergency (red) button, we first unclamp the one.

5. We set the scanner by means of controller computer using CSx software. Required parameters include a set value for the scan resolution in dots per inch (dpi), mode (for now, LLa and LRFB will suffice), gain and heigth. Difference between respective chosen modes embodies in that LRBF excels in sufficiently homogeneous illumination of the entire scanned area of the object from all sides [8] – *left, right, front and back*. LLa provides us in tandem less lighting than LRFB.

   We use by default 600.00 dpi and LLa mode, as these are providing us best correspondence with original work for fine art pieces, and adjust gain and height depending on requirements of the particular artwork we are scanning.

6. Having done all of the previous, we run a preview scan and adjust artwork measurements on controllor computer and check for the result of set parameters.

Once we are satisfied with the preview scans, we are ready to activate scanning. One scanning takes us approximately 15 minutes for smaller pieces of art.

7. Finally, we transform/convert the TIFF file into PNG or JPG for further processing.

3.3 Nine Artworks of Anatoly Fomenko

In pursuit to fulfil the aim of our work, we will use a choice of nine artworks from the broad production of a renowned Russian mathematician and artist Anatoly Fomenko [4]. They all belong to a collection, named Mathematical Impressions. They were downloaded in graphics format .jpg, total size 3,76MB.

Arrayed by title, the selection is following. Selected artworks can be seen in *Appendix C.*

1. ***A HEAVY TOP DRIFTING IN SPACE*** (MATHEMATICAL IMPRESSIONS, No. 74. 1973. *Hamiltonian mechanics, symplectic geometry.* India ink, pencil, and oil on paper, 30.5 × 42.5 cm)

2. ***A SYSTEM OF SHRINKING NEIGHBORHOODS*** (MATHEMATICAL IMPRESSIONS, No. 105. 1973. *Mathematical analysis, topology.* India ink and color pencil on paper, 30.5 × 42.5 cm)

3. ***CELLULAR SPACES*** (MATHEMATICAL IMPRESSIONS, No. 250. 1970. *Topology.* Oil on art board, 50 × 70 cm)

4. ***COMBINATORIAL CONTRACTION*** (MATHEMATICAL IMPRESSIONS, No. 75. 1973. *Combinatorial topology.* India ink, pencil and oil on paper, 30.5 × 43 cm)

5. ***DEFORMATION OF THE RIEMANN SURFACE OF AN ALGEBRAIC FUNCTION*** (MATHEMATICAL IMPRESSIONS, No. 229. 1983. *Theory of algebraic functions.* India ink on paper, 44 × 62 cm)

6. ***SIMPLICIAL SPACES, CELLULAR SPACES, CRYSTAL AND LIQUID*** (MATHEMATICAL IMPRESSIONS, No. 190. 1976. *General ideas in geometry and algebra.* India ink and pencil on paper, 43 × 61.5 cm)

7. ***SINGULARITIES OF SMOOTH FUNCTIONS*** (MATHEMATICAL IMPRESSIONS, No. 180. 1976. *Mathematical analysis and geometry, theory of singularities.* India ink and pencil on paper, 31.5 × 44 cm)

8. ***THE SEPARATIX DIAGRAM OF A CRITICAL SADDLE POINT OF A SMOOTH FUNCTION ON A 3-DIMENSIONAL MANIFOLD*** (MATHEMATICAL IMPRESSIONS, No. 145. 1974. *Analysis on manifolds and vector fields.* India ink and pencil on paper, 35 × 50 cm)

9. ***2-DIMENSIONAL POLYHEDRA AND INCIDENCE MATRICES*** (MATHEMATICAL IMPRESSIONS, No. 174. 1975. *Combinatorial topology.* India ink and pencil on paper, 44 × 61.5 cm)

As a whole, my choice as an author of the work of Anatoly Fomenko‘s artworks was motivated in large measure personally. They felt to me familiar, and I realised this shows exactly how I do define a thought greatness of an artist; his exposal is a very authentic form of inter-human communication, and the outcome is an emotion triggered in the one, with whom the interaction is going off. No matter what this emotion will be, it is accepted, and it offers one a sense of freedom and acceptance.

They hit my interest by intertwistment of detailed structure, associating me with a playful variation of technical form and its pure character, together with instants like spilled grumous black liquid on *2-DIMENSIONAL POLYHEDRA AND INCIDENCE MATRICES*, dispirited eclipsed man on *A SYSTEM OF SHRINKING NEIGHBORHOODS*, surreally, yet familiarly incurvated extents on *COMBINATORIAL CONTRACTION*, almost speaking for a dissociation, and a man running on *SIMPLICIAL SPACES, CELLULAR SPACES, CRYSTAL AND LIQUID*, yet heavily moving comparably to giving one's hardest try into in the long run nothing, and staying stuck in sleep paralysis.

I am absorbed in author‘s sense for accuracy, as well as an ambivalence between subtlest beauties present in what mathematics describes, and sadness caused by intensity, as the intensity may become a dividing line toward separation.

3.4 Nine Artworks of Alexandra Dyalee

Another author interconnecting mathematics with art used for our thesis will be the author herself – Alexandra Dyalee [13]. We are offering a choice of nine artworks of hers (listed below) belonging to a collection named with Latin title *Ars Rationis: Concordia*, from english *conceptual art* standing for *ars rationis*, and *concordia* for *harmony.* They were downloaded in graphics format .jpg, total size 61,5 MB.

Arranged by Latin title, the selection is following. Selected artworks can be seen in *Appendix B.*

1. ***BEATITUDO PURA*** (ARS RATIONIS: CONCORDIA, No. 01. 2019. Lat. THE PURE HAPPINESS. *Inter-mathematical attraction.* Acrylic on triangle canvas, 50 × 50 cm)/2. Photographed painting. JPG image (6,07 MB).
   Published by ARTISTCLOSEUP [15]. Exhibited in ARTFORUM Bratislava, 05.2023.

2. ***DON AMOR*** (ARS RATIONIS: CONCORDIA, No. 16. 2022. Lat. MR. LOVE. Oil on canvas, 10 × 15 cm). Scanned by CRUSE SYNCHRONTABLE SCANNER, 600.00 dpi, LLa, Gain 0.700, height 1.750 cm. TIFF image (654 MB) exported to JPG (4,18 MB) by 40% compression in GIMP.

3. ***IN VIA*** (ARS RATIONIS: CONCORDIA, No. 13. 2021. Lat. THE PATHWAY. *Bond energy dissociation.* Acrylic on canvas, 15 × 10 cm). Photographed painting. JPG image (8,38 MB).
   Published by ARTISTCLOSEUP [15]. Exhibited in ARTFORUM Bratislava, 05.2023.

4. ***MI...*** (ARS RATIONIS: CONCORDIA, No. 17. 2022. Lat. MINE... Oil on canvas, 10 × 15 cm). Scanned by CRUSE SYNCHRONTABLE SCANNER, 600.00 dpi, LLa, Gain 0.700, height 1.750 cm. TIFF image (654 MB) exported to JPG (3,63 MB) by 40% compression in GIMP.

5. ***NOVA SENTENTIA*** (ARS RATIONIS: CONCORDIA, No. 14. 2022. Lat. VIEWPOINT OF FRESHENED. *Permutation of new and old information.* Tempera on black canvas board, 15 × 15 cm). Scanned by CRUSE SYNCHRONTABLE SCANNER, 600.00 dpi, LLa, Gain 0.700, height 0.300 cm. TIFF image (203 MB)

exported to JPG (6,53 MB) by 40% compression in GIMP.
Exhibited in ARTFORUM Bratislava, 05.2023.

6. ***NUMERI NEXUM*** (ARS RATIONIS: CONCORDIA, No. 10. 2020. Lat. FIGURE OF LOVE. *Field of love.* Acrylic on canvas, $50 \times 50 \times 50\sqrt{2}$ cm). Photographed painting. JPG image (4,47 MB).
   Commissioned to Florida. Featured in ARTISTCLOSEUP [16]. Exhibited in ARTFORUM Bratislava, 05.2023.

7. ***PUERITIA*** (ARS RATIONIS: CONCORDIA, No. 17. 2023. Lat. Childhood. *Boundlessness is a function of time.* Oil on canvas, 20×50 cm). Scanned by CRUSE SYNCHRONTABLE SCANNER, 600.00 dpi, LLa, Gain 0.850, height 1.700 cm. TIFF image (316 MB) exported to JPG (9,16 MB) by 40% compression in GIMP.

8. ***TIBI CREDO*** (ARS RATIONIS: CONCORDIA, No. 16. 2023. Lat. I TRUST YOU. *The upper dimension of life.* Acrylic on canvas, 60×30 cm). Scanned by CRUSE SYNCHRONTABLE SCANNER, 600.00 dpi, LLa, Gain 0.900, height 3.800 cm. TIFF image (674 MB) exported to JPG (12,9 MB) by 40% compression in GIMP.
   Commissioned to Manchester. Exhibited in ARTFORUM Bratislava, 05.2023.

9. ***TRIBUS*** (ARS RATIONIS: CONCORDIA, No. 13. 2022. Lat. THE TRIPLET. *1 + 1 = 3.* Acrylic on canvas, 10×15 cm). Scanned by CRUSE SYNCHRONTABLE SCANNER, 600.00 dpi, LLa, Gain 0.700, height 1.500 cm. TIFF image (110 MB) exported to JPG (6,12 MB) by 40% compression in GIMP.

Artworks of *Ars Rationis: Concordia* are in tandem expressing some personal or interpersonal happenings, which might stand for an emotional conflict, unheard internal drama or love, through a definition of **energetical bunches** (these are representing some elements) and **interactions** between energetical bunches (functions of elements), drawing up a mathematical description of what is happening. Taken this way, we are working with mathematics as with a logical tool, that, well understood, can become our key for understanding a nucleus of happenings we are encountering intuitively, moreover coming hand in hand with a representation in physics.

Scale of notation is following: Informal description of the first definition does stand for answering a question: Yes, it is true that everything is energy. But *which one* this is? And here, we find ourselves with an answer.

Definition 3.3. *Bunch of energy.*
Denote $\beta_{\mathbb{X}}$ an unity consisting of a set of k amounts of energy of particular form [5], which are bonded into one system, while each of the (1) forms of energy and (2) bonds of some forms of energy is protected by a respective requirement of energy. Index $\mathbb{X}$ itself will be a label for this particular bunch.

We don't further specify particular amounts of particular energy forms that are bonded in respective bunches, we just suppose there are some, and the subsequent purpose of mathematics present in *Ars Rationis: Concordia* works is to spark the basal emotion shown in the artistic portrayal by a mathematical description of the behaviour of underlying process.

Further, except for the bunches themselves, we are using mainly interactions between them in the meta-language of series *Ars Rationis: Concordia.* For the sake of our work, however, above indication for an intuitive image be to acquired by a reader is sufficient.

3.5 Miscellaneous from the Faculty

Except for the above described, we will also run a fractal analysis on some selected interesting artworks of heterogenous traits and from various authors, with common functioning on the Faculty of Mathematics, Physics and Informatics of CU in Bratislava.

The first artwork of this group will be a portrait of professor Pavol Zlatoš painted in 1980 by Jakub Konarzewski for a student theatre Pegasník. More particular, the artwork meets below metadata, with its portrayal inclosed in *Appendix 3.*

1. ***PROFESSOR OF LOGIC*** (Jakub Konarzewski, 1980. Pastel on recycled paper, 50 x 50 cm). Scanned by CRUSE SYNCHRONTABLE SCANNER, 600.00 dpi, LLa, Gain 1.00, height 0.500 cm. TIFF image (908 MB) exported to JPG (8 MB) by 33% compression in GIMP.

Note. The permission to scan the original by J. Konarzewski was given by the original owner.

Among the digitalized original portrait of professor Zlatoš painted by Konarzewski, we will run fractal analysis on one more interesting version of this picture. To be clear, we are going to use a section of this artwork applied to a front of the book *Ani matematika si nemôže byť istá sama sebou* [14] written by professor Zlatoš.

2. ***NOT EVEN THE MATHEMATICS CAN BE SURE*** (Jakub Konarzewski, 2007. Digitally processed drawing). Downloaded from https://www.databazeknih.cz/. JPG image (32,0 kB).

Another artworks of miscellaneous will be three drawings of two daughters of docent Pavel Chalmovianský from the KAG Section of Geometry. Authors of the three inclosed artworks are Michala Chalmovianská (2x) and Adela Chalmovianská (1x).

3. ***MONSTERY LOVE*** (Michala Chalmovianská, 2023. Drawing). Photographed by smartphone. JPG image (1,95 MB).

4. ***CARD WONDERLAND*** (Michala Chalmovianská, 2023. Drawing). Photographed by smartphone. JPG image (2,50 MB).

5. ***RAINBOW GIRL*** (Adela Chalmovianská, 2023. Drawing). Photographed by smartphone. JPG image (2,37 MB).

4 WORK IN (PROGRESS) PROCESS

4.1 Software Specifications

The intended future of the software we will be using to maintain results is being able to load the artwork data (in a form of *.jpg* image), convert it into its binary alternative and calculate its fractal dimension in respect of the box counting algorithm. The realisation in the terms of hardware, software and interface should look following.

Communicational interface comprises of the functions *Open*, *Make Binary* and *Fractal Box Count*, additionally we may require functions *Demo* and *Help*. Further, we allow to step back arbitrarily many times with function *Undo* and execute the working window by clicking the red cross icon.

In particular,

- running the function *Open* redirects the user into one's file storage allowing to select an arbitrary *.jpg* image, which will be uploaded into program after double-clicking it,
- running the function *Make Binary* works only if there is uploaded an image and converts the 24-bit RGB color palette of the image into the true black and white color palette, that uses 1-bit color which yields two different colors of black and white,
- running the function *Fractal Box Count* calculates and displays obtained fractal dimension of respective binary image data together with a respective log / log graph, using the box counting method.
- function *Demo* should open and replay a brief video manual for using the software, possible to be started at chosen point, paused or executed at any moment,
- function *Help* opens a pop-up window informing the user about next possible steps of software usage bound to the actual step in which the Help was clicked on,
- clicking *Undo* reloads the software into the phase it was at before performing the latest action by the user, and
- clicking *Info* opens a pop-up window containing basic information about the software product and its author.

4.2 Software FracLac for ImageJ

For verification of the intended implementation, we started experimenting with FracLac. Picking up on [11], we are switching our attention to an open-source software ImageJ, which is developed apart from the other for calculation of the Fractal Box Count.

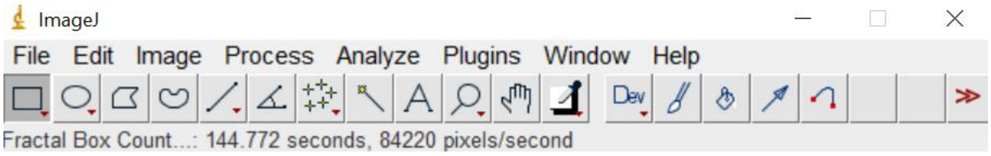

Figure 5: ImageJ (main menu)

The software is written in Java, allowing it to run on Linux, Mac OS X and Windows, in both 32-bit and 64-bit modes. ImageJ opens and saves all supported data types as TIFF (uncompressed) or as raw data.

The process of calculating the desired fractal dimension consists of such:

- We double-click on the software icon and choose FILE > OPEN..., then, we choose a file of corresponding type (we will be prefering .jpg for that all of our data is of uniform format).
- We transform the file into a binary picture. We choose PROCESS > BINARY > MAKE BINARY. The original picture will change into black-or-white binary picture. (See specimen transformation of the painting Beatitudo Pura (Dyalee) below.)
- We run the Fractal Box Count function on obtained binary image. Particularly, we proceed on ANALYZE > TOOLS > FRACTAL BOX COUNT. At this point, software asks us to fill the parameters of Box Sizes into dialogue window. For now, we will proceed with default values (see Figure 7).

In the simplest terms, the software counts the number of boxes of a given size needed to cover a one pixel wide, binary (black on white) border. [12] The procedure is by default repeated for boxes that are 2 to 64 pixels wide.

4.3 Experiments in ImageJ

Now, we proceed to execute three experiments. First, we compare following groups of paintings, divided on the basis of **performance time**:

- a simple sketch or paint, with performance time **under 30 min of drawing**,

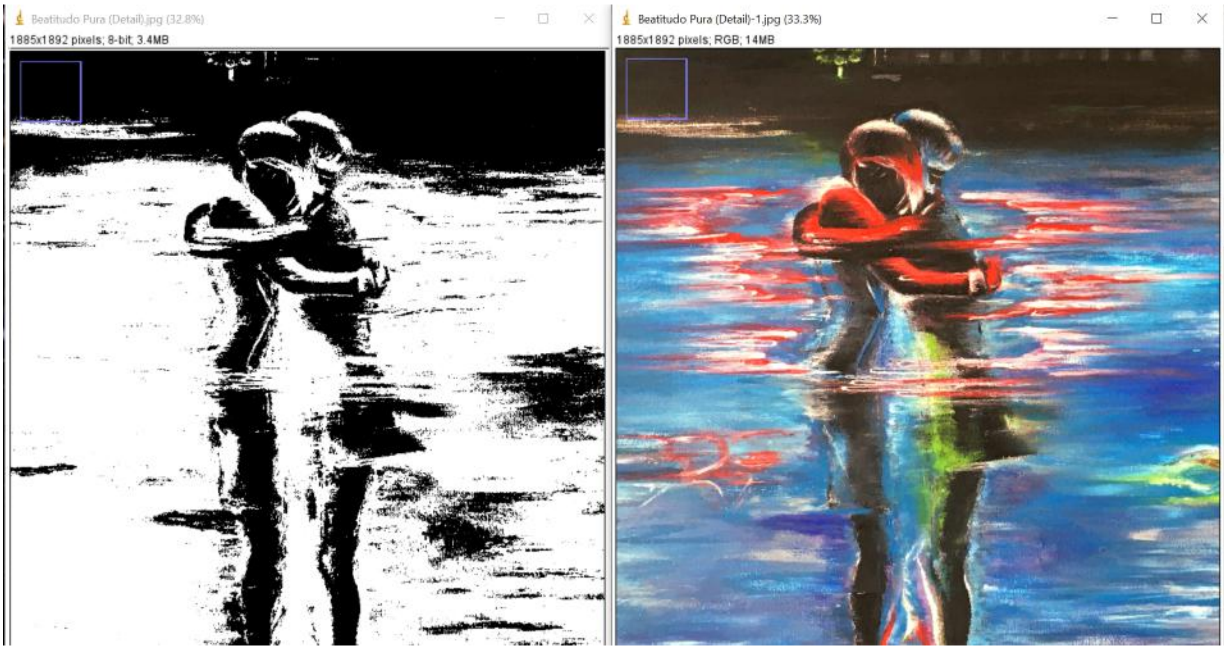

Figure 6: Binary 8-bit image vs. RGB image

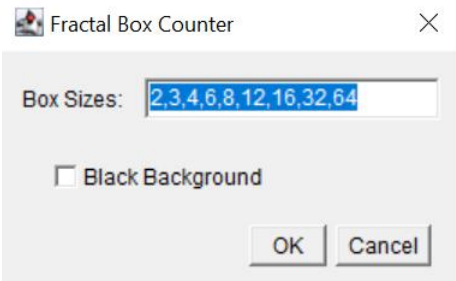

Figure 7: Fractal Box Counter

- mediumly elaborated painting, with performance time **under 10 hours of painting**,
- elaborated painting, with performance time **50 or more hours of painting,**

and then, we compare fractal dimensions between groups of elaborated paintings of different authors and open up a question about possible impact of set box sizes on final results.

EXPERIMENT 1. First, we are going to work with artworks of Dyalee. We use three of the simple sketches or paints, three mediumly elaborated paintings and three elaborated paintings.

Three simple sketches will include *The Beach* (5520 x 3624 pixels), *In The Dark* (4808 x 4608 pixels) and *Sopot* (4568 x 6344 pixels) (see Appendix 3). Three mediumly elaborated paintings will include *Don Amor* (3010 x 4851 pixels), *In Via* (11500 x 8000 pixels) and *The Tribus* (2662 x 4145 pixels) (see Appendix 1), and three elaborated paintings will include *Beatitudo Pura* (3000 x 3000 pixels), *Tibi Credo* (14003 x 6930 pixels) and *Numeri Nexum* (5926 x 8000 pixels) (see Appendix 1). Measured fractal dimensions of such artworks transformed into binary versions are following.

	Label	C2	C3	C4	C6	C8	C12	C16	C32	C64	D
1	TheBeach	4447740	1992204	1127102	505326	286147	128543	73063	18808	4839	1.969
2	InTheDark	5203210	2325656	1312895	587628	331881	148940	84427	21555	5469	1.978
3	Sopot	6857295	3072652	1737823	779991	440498	197555	111846	28329	7198	1.979
4	DonAmor	2884263	1305032	743739	336690	192035	86698	49489	12727	3301	1.955
5	InVia	17566427	7941696	4521436	2052216	1173557	535865	307370	81156	21417	1.935
6	Tribus	2208278	994548	565936	256087	146712	67009	38785	10338	2697	1.932
7	BeatitudoPura	1063000	490705	284938	132959	77483	36309	21441	5943	1684	1.861
8	TibiCredo	14115407	6565085	3820348	1790050	1048852	493157	288379	78098	21089	1.874
9	NumeriNexum	5615428	2615367	1522011	720402	425649	204743	121921	34896	9748	1.828

Figure 8: Fractal Dimensions of Differently Elaborated Artworks

We see there are spectable differences between respective three groups of artworks. We will interpret them further in Chapter 5.1.

EXPERIMENT 2. We will compare fractal dimensions of nine selected artworks of Alexandra Dyalee and Anatoly Fomenko, respectively (see 3.3, resp. 3.4) with box sizes set $\{1, 2, 4, 8, 16, 32, 64\}$. Measured fractal dimensions of these 18 artworks transformed into binary versions are following.

	Label	C2	C3	C4	C6	C8	C12	C16	C32	C64	D
1	(D)BEATITUDOPURA	1063000	490705	284938	132959	77483	36309	21441	5943	1684	1.861
2	(D)DONAMOR	2884263	1305032	743739	336690	192035	86698	49489	12727	3301	1.955
3	(D)INVIA	17566427	7941696	4521436	2052216	1173557	535865	307370	81156	21417	1.935
4	(D)MI...	2368957	1160172	699271	337782	199559	92960	53618	13977	3625	1.872
5	(D)NOVASENTENTIA	7970963	3757937	2206389	1046320	617789	294659	173678	47555	12503	1.856
6	(D)NUMERINEXUM	5615428	2615367	1522011	720402	425649	204743	121921	34896	9748	1.828
7	(D)PUERITIA	10888379	5024213	2895332	1327799	762447	347897	199050	51561	13196	1.937
8	(D)TIBICREDO	14115407	6565085	3820348	1790050	1048852	493157	288379	78098	21089	1.874
9	(D)TRIBUS	2208278	994548	565936	256087	146712	67009	38785	10338	2697	1.932

Figure 9: Fractal Dimensions of Selected Artworks (Dyalee)

	Label	C2	C3	C4	C6	C8	C12	C16	C32	C64	D
1	(F)2-DIMENSIONALPOLYHEDRA...	117262	57766	33269	14900	8449	3762	2160	540	140	1.958
2	(F)AHEAVYTOPDRIFTINGINSPACE	126571	57227	32538	14620	8331	3736	2154	540	140	1.966
3	(F)ASYSTEMOFSHRINKINGNEIGHBORHOODS	121569	54781	31093	13881	7870	3544	1997	512	140	1.960
4	(F)CELLULARSPACES	97317	44818	25794	11940	6857	3156	1832	491	139	1.896
5	(F)COMBINATORIALCONTRACTION	112755	51826	29834	13695	7902	3616	2083	540	140	1.930
6	(F)DEFORMATIONOFTHERIEMANNSURFACEOFANALGEBRAICFUNCTION	174522	85091	49760	23014	13097	5952	3379	849	221	1.937
7	(F)SIMPLICIALSPACES;...	97615	46338	26919	12399	7081	3228	1838	475	130	1.922
8	(F)SINGULARITIESOFSMOOTHFUNCTIONS	95801	44808	26141	12109	7040	3254	1889	518	139	1.886
9	(F)THESEPARATIXDIAGRAM...	299804	135497	77040	34723	19726	8924	5076	1325	359	1.946

Figure 10: Fractal Dimensions of Selected Artworks (Fomenko)

Highlighted lines correspond to greatest measured values of selections of respective authors.

EXPERIMENT 3. To verify extrinsically what is the impact of set box sizes on a result, we use an artwork from EXPERIMENT 2 with the highest measured fractal dimension (A HEAVY TOP DRIFTING IN SPACE), and make 10 measurements: first nine are with box size sets $\{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \ldots, \{1,2,3,4,5,6,7,8,9,10\}$. In the last one, we add a number 1000 of box size to see whether adding a bigger number have an impact on the result. Results are following.

	Label	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C1000	D
1	(F)AHEAVYTOPDRIFTINGINSPACE	493847	126571	57227	32538	20757	14620	10799	8331	6577	5304	1	1.893
2	(F)AHEAVYTOPDRIFTINGINSPACE	493847	126571	57227	32538	20757	14620	10799	8331	6577	5304	0	1.967
3	(F)AHEAVYTOPDRIFTINGINSPACE	493847	126571	57227	32538	20757	14620	10799	8331	6577	0	0	1.965
4	(F)AHEAVYTOPDRIFTINGINSPACE	493847	126571	57227	32538	20757	14620	10799	8331	0	0	0	1.965
5	(F)AHEAVYTOPDRIFTINGINSPACE	493847	126571	57227	32538	20757	14620	10799	0	0	0	0	1.966
6	(F)AHEAVYTOPDRIFTINGINSPACE	493847	126571	57227	32538	20757	14620	0	0	0	0	0	1.966
7	(F)AHEAVYTOPDRIFTINGINSPACE	493847	126571	57227	32538	20757	0	0	0	0	0	0	1.966
8	(F)AHEAVYTOPDRIFTINGINSPACE	493847	126571	57227	32538	0	0	0	0	0	0	0	1.962
9	(F)AHEAVYTOPDRIFTINGINSPACE	493847	126571	57227	0	0	0	0	0	0	0	0	1.962
10	(F)AHEAVYTOPDRIFTINGINSPACE	493847	126571	0	0	0	0	0	0	0	0	0	1.964

Figure 11: Expansion of Box Sizes Set

Highlighted lines correspond to the most incident value.

5 DID WE SUCCEED?

Now it comes to by far the most important question of the work. Did we find an answer to at least one of the given inquiries?

Q1: **What is an "usual" fractal dimension of an artwork?**

Q2: **Does it depend upon the author?**

Q3: **Is there a difference between fractal dimensions of differently elaborated artworks[1]?**

Q4: **What is the particular number of fractal dimension revealing about the artwork?**

We will untack this by parts, aiming to provide both extrinsic and intrinsic viewpoints.

5.1 Experiments Interpretation

EXPERIMENT 1. We loaded nine artworks (listed in 4.3), of which *Beatitudo Pura* and *Numeri Nexum* were not scanned by Cruse Synchrontable Scanner (3.2), while the other seven were. We divided the group of nine artworks into three subgroups: simple sketches or paints (*#1 The Beach*, *#2 In the Dark*, *#3 Sopot*; mediumly elaborated paintings (*#4 Don Amor*, *#5 In Via*, *#6 Tribus*) and elaborated paintings (*#7 Beatitudo Pura*, *#8 Tibi Credo*, *#9 Numeri Nexum*).

Fractal dimensions of the selected nine paintings did range between 1.828 (Numeri Nexum) and 1.979 (Sopot), while held following: **Ranged by D, the three groups of respective elaborations remained unchanged.** That means, sketches or simple paintings obtained greatest fractal dimensions (1.969-1.979), mediumly elaborated painting fractal dimensions were in the middle (1.932-1.955) and elaborated paintings obtained lowest fractal dimensions (1.828-1.861) in this experiment. **Ordering the artworks by table row index #i** (Figure 8), hold $\forall i \in \{1, 2, \ldots, 9\} : \#i \leq \#j \Rightarrow D(\#i) \leq D(\#j)$. Considering these results, we can deduce following remark:

Remark. *According to analyzed data in EXPERIMENT 1, we state fractal dimension of elaborated paintings obtain lowest values, fractal dimensions of mediumly elaborated paintings obtain larger values compared to elaborated paintings, and simple sketches and paints obtain largest values of fractal dimension.*

Important conclusion is that ***there is a difference of fractal dimensions between differently elaborated paintings.***

[1]Measure of elaboration in an artwork is our novel image feature, picking up on *proportion*, *originality*, *perspective*, *intention* and *history with significance* [15].

Note. *Lower values of fractal dimension in elaborated paintings might be a result of that two artworks of that group are not scanned in optimal highest possible quality with Cruse Synchrontable Scanner (600 dpi). However, this is only our possible hypothesis, which has not been inquired further in this work.*

EXPERIMENT 2. We loaded 18 artworks, of which 9 are created by Dyalee and 9 are created by Fomenko. Denote their measured fractal dimensions $D_{D1}, D_{D2}, \dots, D_{D9}$, for Dyalee's artworks, and $D_{F1}, D_{F2}, \dots, D_{F9}$ for Fomenko's artworks. We were interested in whether there is anything the measured values of fractal dimensions could disclose about respective authors' styles.

First, we compare the **central tendency** of measured fractal dimensions of artworks of those two authors. This includes the terms of mean, median and mode. Mean of Fomenko's production is $\overline{D_F} = \frac{\sum_{i=1}^{9} D_{Fi}}{9} \approx 1.933$, for Dyalee, it is $\overline{D_D} = \frac{\sum_{i=1}^{9} D_{Di}}{9} \approx 1.894$.

In the terms of median, Fomenko's artworks range around value $D_{F5} = 1.937$, while Dyalee's artworks range around $D_{D5} = 1.874$. For both groups of artworks, there is no two with same measured dimension, so we exclude terms of mode.

Second, we compare **variability** of the two respective data sets. This comprises of range, standard deviation and variance.

Range of fractal dimensions $Range(D) = D_{max} - D_{min}$ holds $Range(D_D) = D_{D(max)} - D_{D(min)} = 0.127$ for Dyalee, while it is equal to $Range(D_F) = D_{F(max)} - D_{F(min)} = 0.070$ for Fomenko. We can see **Dyalee's artwork fractal dimensions are of larger range, that may be caused by varying levels of elaboration**, as shown in EXPERIMENT 1. Mean absolute deviation $MD = \frac{\sum_{i=1}^{9} |D_i - \overline{D}|}{9}$ for Fomenko's artworks is $MD_F = \frac{\sum_{i=1}^{9} |D_{Fi} - \overline{D_F}|}{9} = 0.022$, while it is $MD_D = \frac{\sum_{i=1}^{9} |D_{Di} - \overline{D_D}|}{9} = 0.040$ for Dyalee's artworks. Variance σ^2 for Fomenko's artworks is $\sigma_F^2 = \frac{\sum_{i=1}^{9} (D_{Fi} - \overline{D_F})^2}{9} = 0.03$, while it is $\sigma_D^2 = \frac{\sum_{i=1}^{9} (D_{Di} - \overline{D_D})^2}{9} = 0.12$ for Dyalee's artworks.

Of such analysis[2], we can conclude **the style of Fomenko's artworks selected for these experiments is "more fractal" by both mean and median, and also more consistent by range, standard deviation and variance**.

EXPERIMENT 3. To investigate for an optimal selection of box sizes, we made an experiment comparing fractal dimensions in respect of increasing the box sizes from size set $\{1, 2\}$ to the box size set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Finally, we tried to add one significantly larger box size (1000) to see whether this will impact the results.

Experiment showed us that for box size sets $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 4, 5, 6\}$ and $\{1, 2, 3, 4, 5, 6, 7\}$, the fractal dimension D is same, equal to the value of $D \approx 1.966$. This is also

[2]For statistical analysis, we used modular methods taught in compulsory courses of Probability and Statistics.

the most incident value (mode) of this measurement. Excluding the case #1, where we added into consideration a box of size 1000, values of D deflect marginally from the most frequent case. Largest deflection from this value is in cases #8 and #9, which is -0.004.

Adding up a terminology of **expected value**, cases #2 − #9 mediate 1.9645 ± 0.0025. Therefore, we state **for this data set, difference between fractal dimensions measured on box size sets** $\{1, 2\}, \ldots, \{1, 2, \ldots, 10\}$ **is negligible.** Extrinsically viewed, we could suppose the measured fractal dimension with neglectable deflections made by slight changes in box size sets should brings us the most "clear" information about observed data. This result could be interesting for further making of experiments in ImageJ with fine art pieces.

Interesting is to note that as we added a box of size 1000, in case #1, fractal dimension D decreased by 0.073. It provokes one to further examine the running of ImageJ on this case and find out a reason why this is happening, so we could derive an objective statement about optimal selection of box sizes set.

5.2 Questions Answered

Finally, is there any kind of responsible answers we did find regarding our stated questions Q1-Q4?

Interpreting this consequently, we see:

Q1. Measuring fractal dimensions of 23 selected artworks of broad spectrum of features, plus three indescribed sketches used in EXPERIMENT 1, we obtained following set of fractal dimensions (ordered by size): $DIM = \{1.801, 1.824, 1.828, 1.856, 1.858, 1.861, 1.870, 1.872, 1.874, 1.886, 1.896, 1.922, 1.926, 1.930, 1.932, 1.935, 1.937, 1.937, 1.946, 1.955, 1.958, 1.960, 1.966, 1.969, 1.978, 1.979\}$. All of these values can be found in EXPERIMENTS 1, 2, 3, respectively, and a FracLac evaluation table of paintings from the group *Miscellaneous from the Faculty* below.

	Label	C2	C3	C4	C6	C8	C12	C16	C32	C64	D
1	PROFESSORZLATOŠ	7545580	3528464	2067717	973902	571737	270053	158235	43494	12004	1.858
2	AniMatematika	5719	2713	1593	766	441	218	127	34	12	1.801
3	CARDWONDERLAND	1948386	902410	525194	243683	140750	64295	36873	9512	2454	1.926
4	MONSTERYLOVE	1717088	791216	458309	214183	125719	59329	35005	9611	2566	1.870
5	RAINBOWGIRL	1341777	652133	391822	190352	113742	54581	32293	8884	2376	1.824

Figure 12: Fractal Dimensions of Miscellaneous

For the sake of statistical analysis, denote the respective values $\mathcal{D}_1, \ldots, \mathcal{D}_{26}$. Running identical analysis on this set as in Subsection 5.1, we see that **mean** of the set of measured fractal dimensions is $\overline{\mathcal{D}} = \frac{\sum_{i=1}^{26} \mathcal{D}_i}{26} \approx 1.910$. Median of this data set is $Med(Di) = \frac{\mathcal{D}_{13}+\mathcal{D}_{14}}{2} = 1.928$, and mode of the set hold $Mode(Di) = 1.937$.

Considering **variability** of this set, we have a range of $Range(Di) = \mathcal{D}_{max} - \mathcal{D}_{min} = 0.178$. Mean absolute deviation is equal to $MAD(Di) = \frac{\sum_{i=1}^{26} |\mathcal{D}_i - \overline{\mathcal{D}}|}{26} \approx 0.045$ and standard deviation (variance) is equal to $\sigma^2_{Di} = \frac{\sum_{i=1}^{26} (\mathcal{D}_i - \overline{\mathcal{D}})^2}{26} \approx 0.0027$.

Deducing from obtained data what is an "usual", perhaps expected, fractal dimension of an artwork, we can make three conclusions. With respect to data and methods used in our thesis:

1. *Expected fractal dimension of an artwork in the terms of mean is around value 1.910 with mean absolute deviation around 0.045.*
2. *Expected fractal dimension of an artwork in the terms of median is a value of 1.928, while expected range of all fractal dimensions is around value of 0.178.*
3. *In intuitive understanding, reliance of stated expected fractal dimensions is high as variability of the set of all measured fractal dimensions is a small value of 0.0027.*

Q2. Seeing the analysis performed in interpretation of EXPERIMENT 2, we can state in the two particular cases of Dyalee and Fomenko, the expected fractal dimension in the meaning of particular statistical analysis tools does depend upon the author.

Q3. See Remark in the interpretation of EXPERIMENT 1.

Q4. Ascertaining what is the particular number of fractal dimension revealing about the artwork is a very complex question to be answered. Our intuitive observation is however pleasing, as we aimed to investigate the interconnection between art and nature taking account of the fact that many patterns of nature are fractal. With the use of box counting algorithm, we found out that our selected artworks have their fractal dimensions greater or equal to 1.828, which is a "big" measure from the interval [1,2). So we could hypothesize that the fine art artworks could in general have strong fractal features.

Further, we hypothesize the particular fractal dimension could help us **determine the level of elaboration**, potentially **determine the used medium** (as in cases of sketches, all of the measured fractal dimensions were larger than in cases of paintings) and in a rough way **characterize or predict the particular author** by previously having loaded enough data of their production.

6 CONCLUSION

To conclude, we state we successfully found answers on stated questions of our thesis, as well as we made remarks of possible improvements in the aim of making the work more effective and reliable. All of that is mentioned and described in CHAPTER 5 – DID WE SUCCEED?

One part of our research has also been suggested by committee to be published in the Textbook of the Students' Conference on Science (ŠVK), that is held yearly at our home faculty – Faculty of Mathematics, Physics and Informatics CU BA. This mentioned research segment stays for an extended abstract of our thesis, answering on one selected question of the four we work with at all, analysing fractal dimensions of artworks with given levels of elaboration, which is also our novel image feature. You can see the extended abstract attached in APPENDIX A). Author of the work has been awarded a Laureate Diploma for the research segment.

Though, having found out features that might catch one's interest (at least, it did so with ours), we aim to move with the whole thing forward in the future from that point. To be particular, we realised the fractal dimensions we are using as a tool for interpretation of some selected features of fine art artworks are dependent upon the software which is calculating it, together with quality and technical uniformity of data we have available. We made fractal analysis of binary versions of respective images, which is directly dependent upon the color intensity boundary, that turns the pixel into black or white pixel after transformation. Because of that, our results are just approximate, but if we ran the fractal analysis through all of the layers of color – red, green and blue, without a need to neglect some minor details, that we must have done in the binary transformation, we could get a clear and reliable result, possible to be spoken of as plausible rather than approximate. Also, it would be more effective to use other transformations than just binary versions of our images, as is for example segmentation. In aim to acquire this point, we have prepared the software specification necessary for development of our own software, that fulfils all of the aforementioned needs.

Winding it up, in our thesis, we asked whether measures of fractal dimension even can speak for features of pieces of fine arts – and the answer is yes, they perhaps do so. However, aiming to make them speak for these features in a more plausible way taking account of the complexity of fine art pieces, we have to technically improve the range our fractal analysis tools. This can be done in the future by programming a software specified for the requirements of fractal box count of fine art pieces.

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8 APPENDIX

8.1 A.

Extended abstract explaining partial results of the work presented at the Student's Conference on Science at home faculty. With such paper, author obtained the **Laureate Diploma**.

Fractal Dimension of Fine Art Pieces

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Abstract: Conception of fractal geometry has been embellishing our knowing as such since the novelty era of Mandelbrot, in 1980s, yet it is important also to realise the fractal world had been here with us since ever. Also, within living memory, there have existed a group of people, for each era specific, called *artists*. Capturing the prominently deep sentiment, they have been adapting exactly toward the form of unity with nature, further speaking nothing of the cases there was no need to adapt.

For having a prominently singular conjunction on having something to do with *nature*, we decided to examine the interconnection between these two.

Mathematical tools used in such research are all about the fractal geometry of nature; **fractal dimension** on natural patterns and **box counting dimension** providing us the approximate fractal dimension of fine art pieces.

Keywords: fractal geometry, fractal dimension, box counting method, ImageJ

What is Fractal Dimension?

We work with terms of fractal dimension $D = \lim_{\varepsilon \to 0} \left\{ \frac{\ln \mathcal{N}(A,\varepsilon)}{\ln\left(\frac{1}{\varepsilon}\right)} \right\}$, where $A \in \mathscr{H}(X)$ with a metric space (X,d). Further, if we denote $\mathcal{N}_n(A)$ the number of boxes of side length $(\frac{1}{2^n})$ which intersect the attractor, we can prove the fractal dimension D of A is $\lim_{n\to\infty} \left\{ \frac{\ln \mathcal{N}_n(A)}{\ln 2^n} \right\}$ [Weisstein,].

Problems

(P1) What is an "usual" fractal dimension of a fine-arts artwork? Does it depend upon the author?

(P2) Is there a difference between fractal dimensions of differently elaborated artworks?

(P3) Is the particular number of fractal dimension revealing a statement about some quality attribute of the artwork?

Selected Results

Problem 2: We divided the selection of artworks of one author into three groups: simple sketches or paints (1-3), mediumly elaborated paintings (4-6) and elaborated paintings (7-9). Using software ImageJ [Abramoff, M.D., Magalhaes, P.J., Ram, S.J.] for fractal analysis with box sizes $\{2,3,4,6,8,12,16,32,64\}$ (necessary for the load of box counting), we obtained three intervals of fractal dimensions (see Figure).

	Label	C2	C3	C4	C6	C8	C12	C16	C32	C64	D
1	TheBeach	4447740	1992204	1127102	505326	286147	128543	73063	18808	4839	1.969
2	InTheDark	5203210	2325656	1312895	587628	331881	148940	84427	21555	5469	1.978
3	Sopot	6857295	3072652	1737823	779991	440498	197555	111846	28329	7198	1.979
4	DonAmor	2884263	1305032	743739	336690	192035	86698	49489	12727	3301	1.955
5	InVia	17566427	7941696	4521436	2052216	1173557	535865	307370	81156	21417	1.935
6	Tribus	2208278	994548	565936	256087	146712	67009	38785	10338	2697	1.932
7	BeatitudoPura	1063000	490705	284938	132959	77483	36309	21441	5943	1684	1.861
8	TibiCredo	14115407	6565085	3820348	1790050	1048852	493157	288379	78098	21089	1.874
9	NumeriNexum	5615428	2615367	1522011	720402	425649	204743	121921	34896	9748	1.828

Simple sketches or paints mediates between $D_1 \approx 1.969$ and $D_3 \approx 1.979$. Mediumly elaborated paintings mediate between $D_6 \approx 1.932$ and $D_4 \approx 1.955$. Finally, elaborated paintings mediate between $D_9 \approx 1.828$ and $D_8 \approx 1.874$.

Aim for interpretation of the results drives us toward many more questions. First, we see there *is* a fractal difference present in respective levels of elaboration, but the order of elaboration levels compared to the height of fractal dimension might be considered the counter-intuitive one. Such factor, together with other implications of our results, may as well attract our fancy, but further would meaningfully drive one toward a successful intrinsic continuation of this research.

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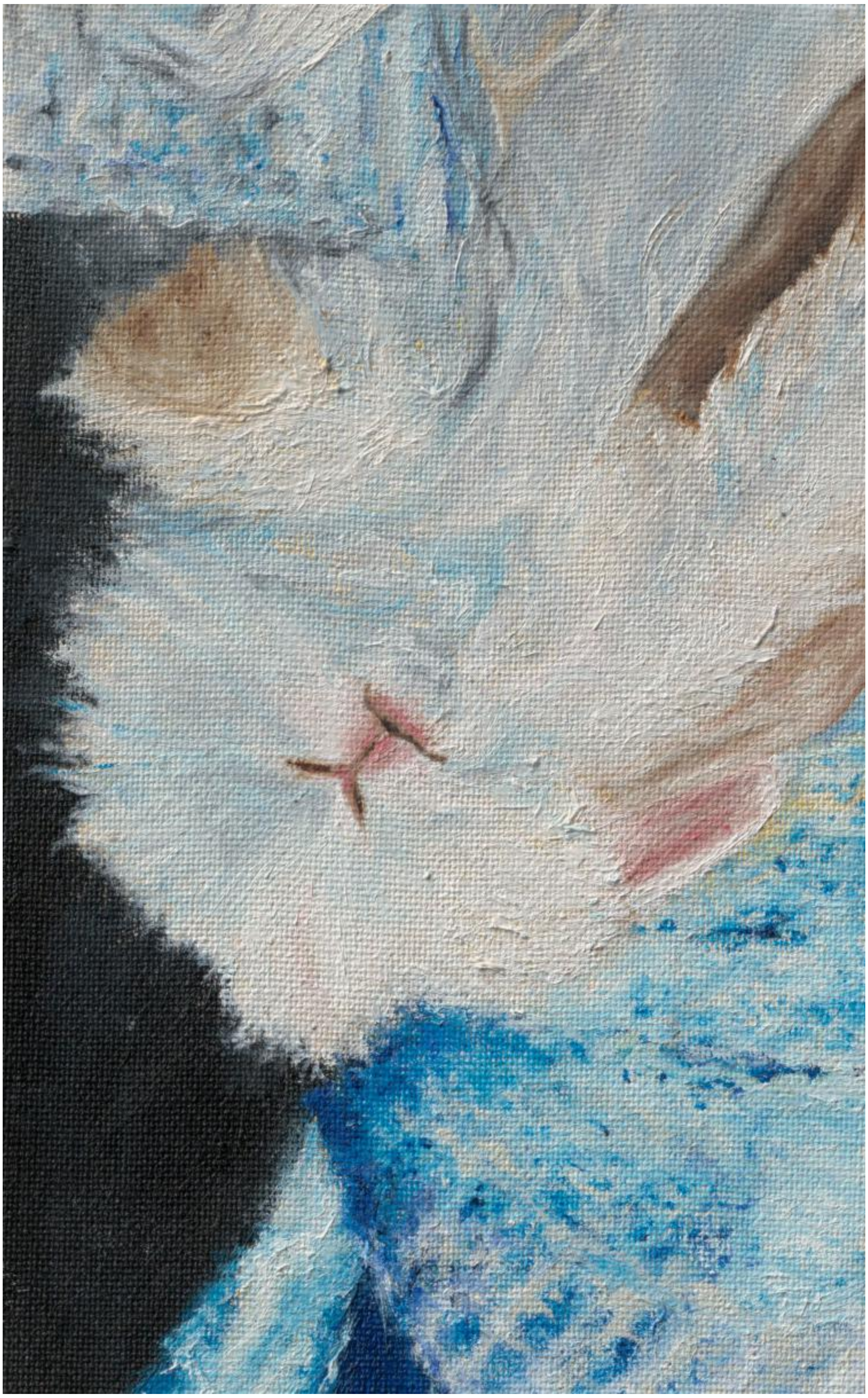
8.2 B.

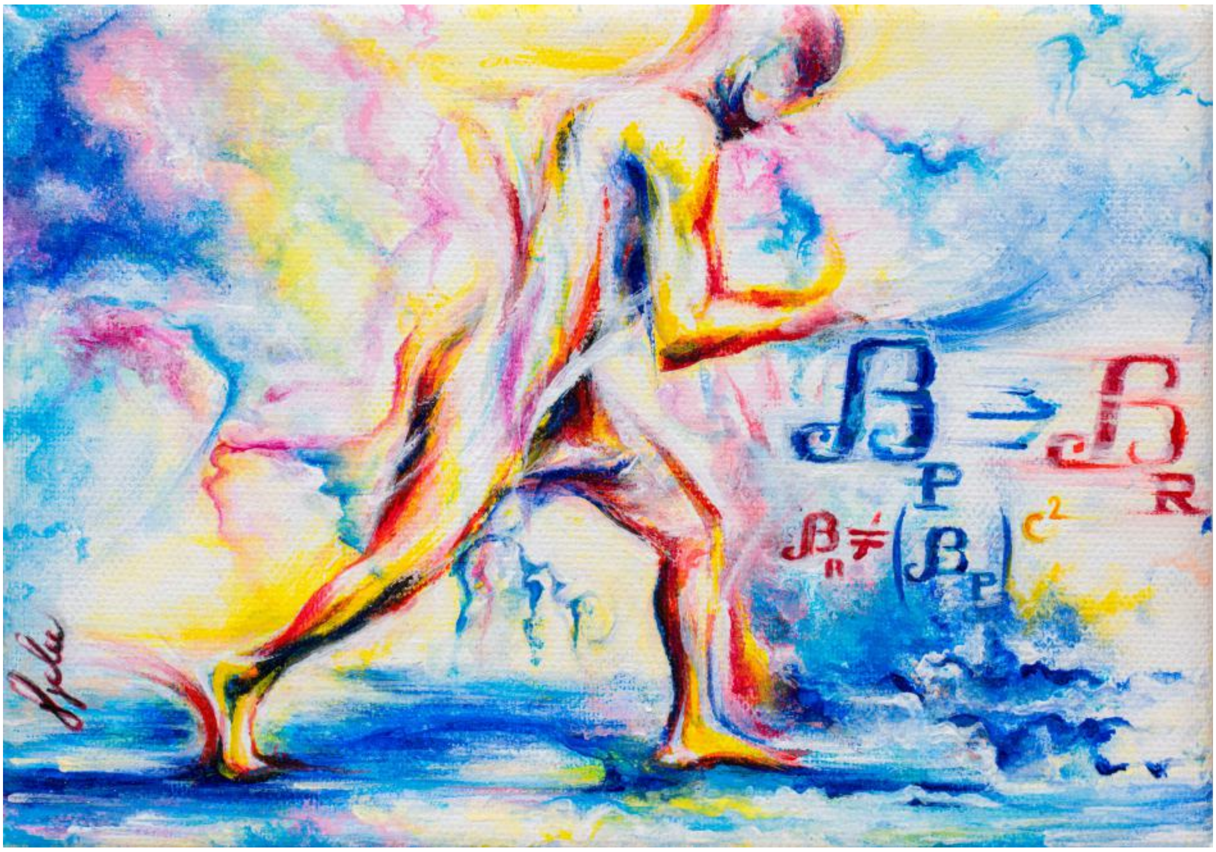
Nine artworks of Alexandra Dyalee.

1. ***BEATITUDO PURA***

2. ***DON AMOR***



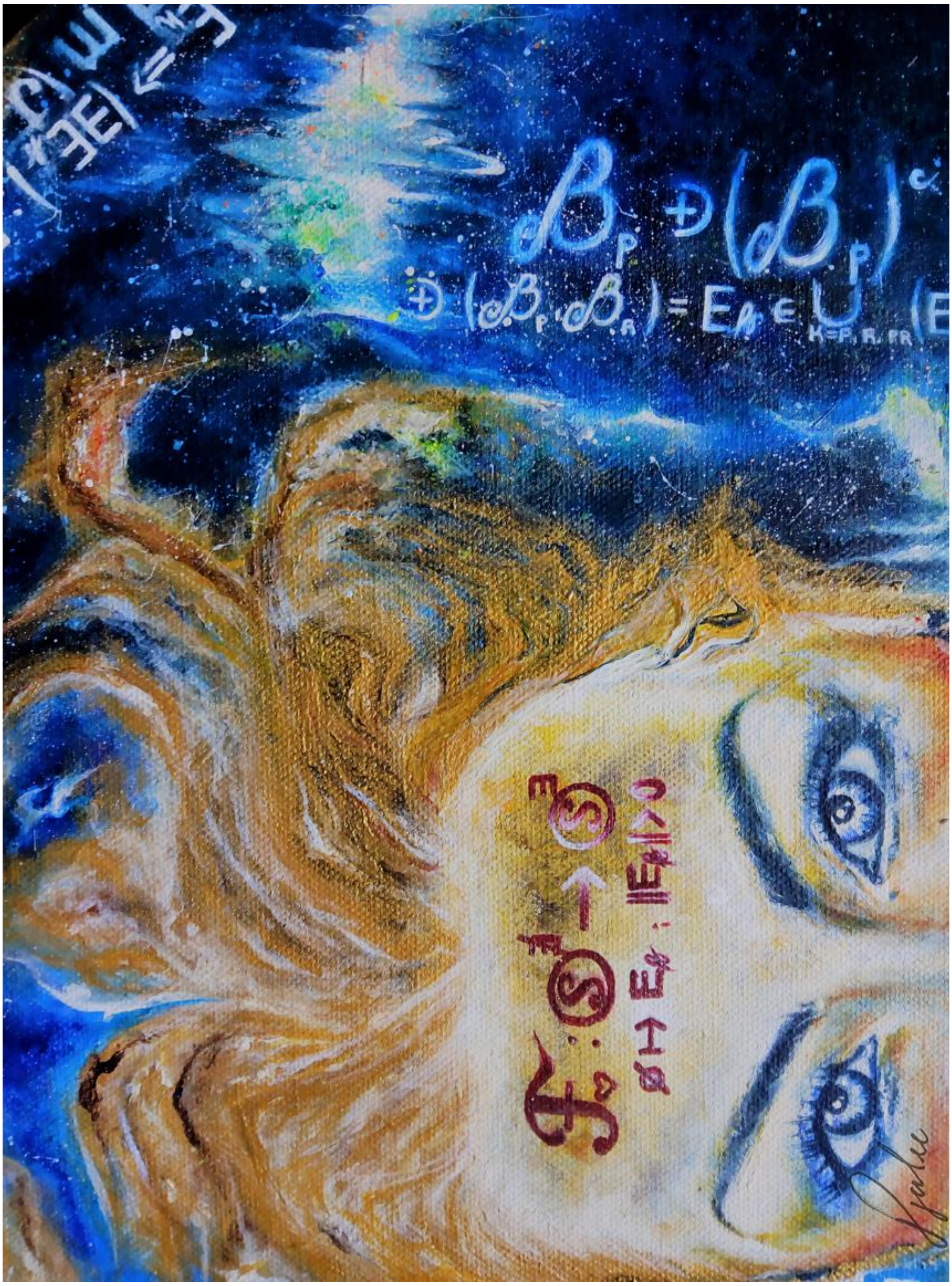
3. ***IN VIA***



4. ***MI...***

5. ***NOVA SENTENTIA***

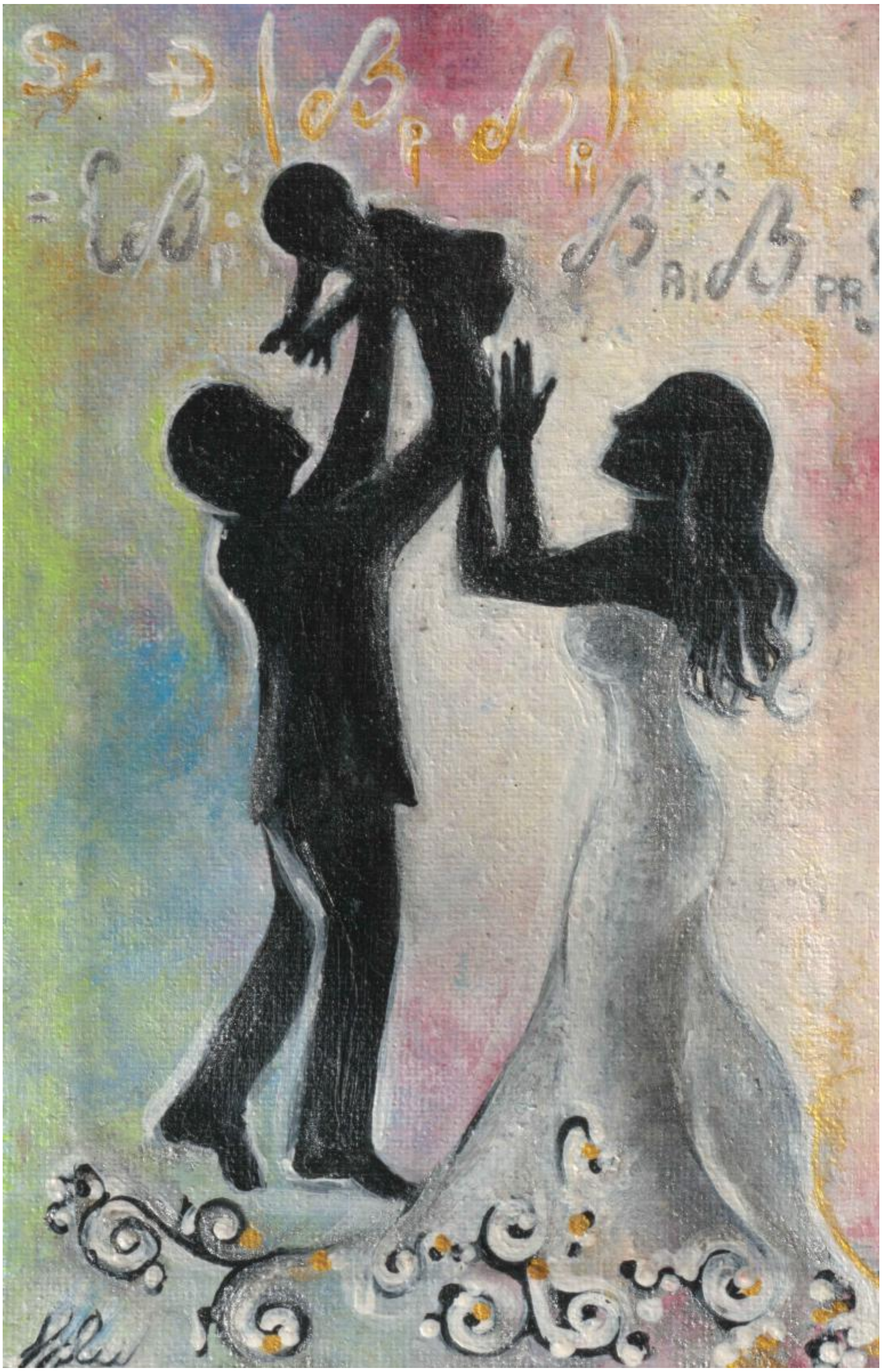
6. ***NUMERI NEXUM***



7. *PUERITIA*

8. *TIBI CREDO*

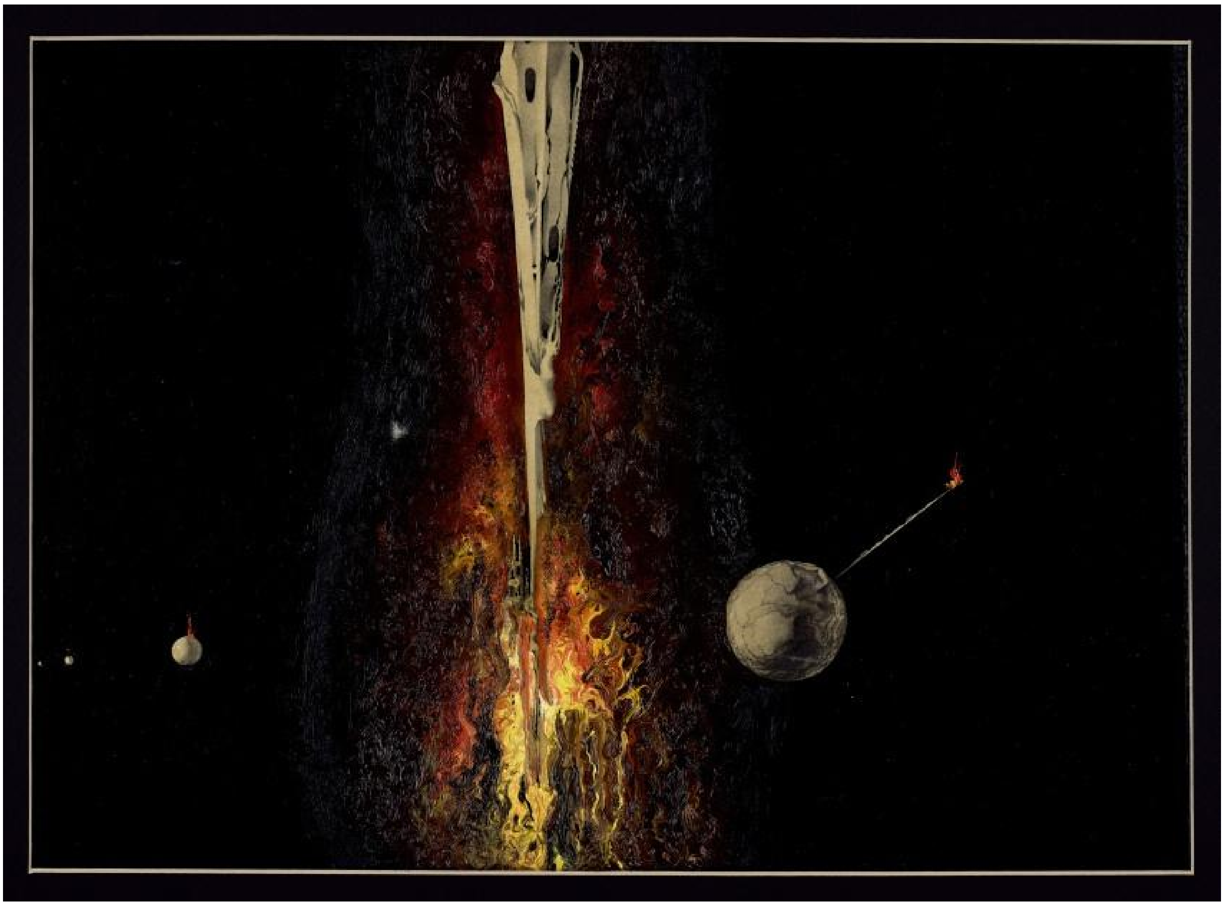
9. ***TRIBUS***



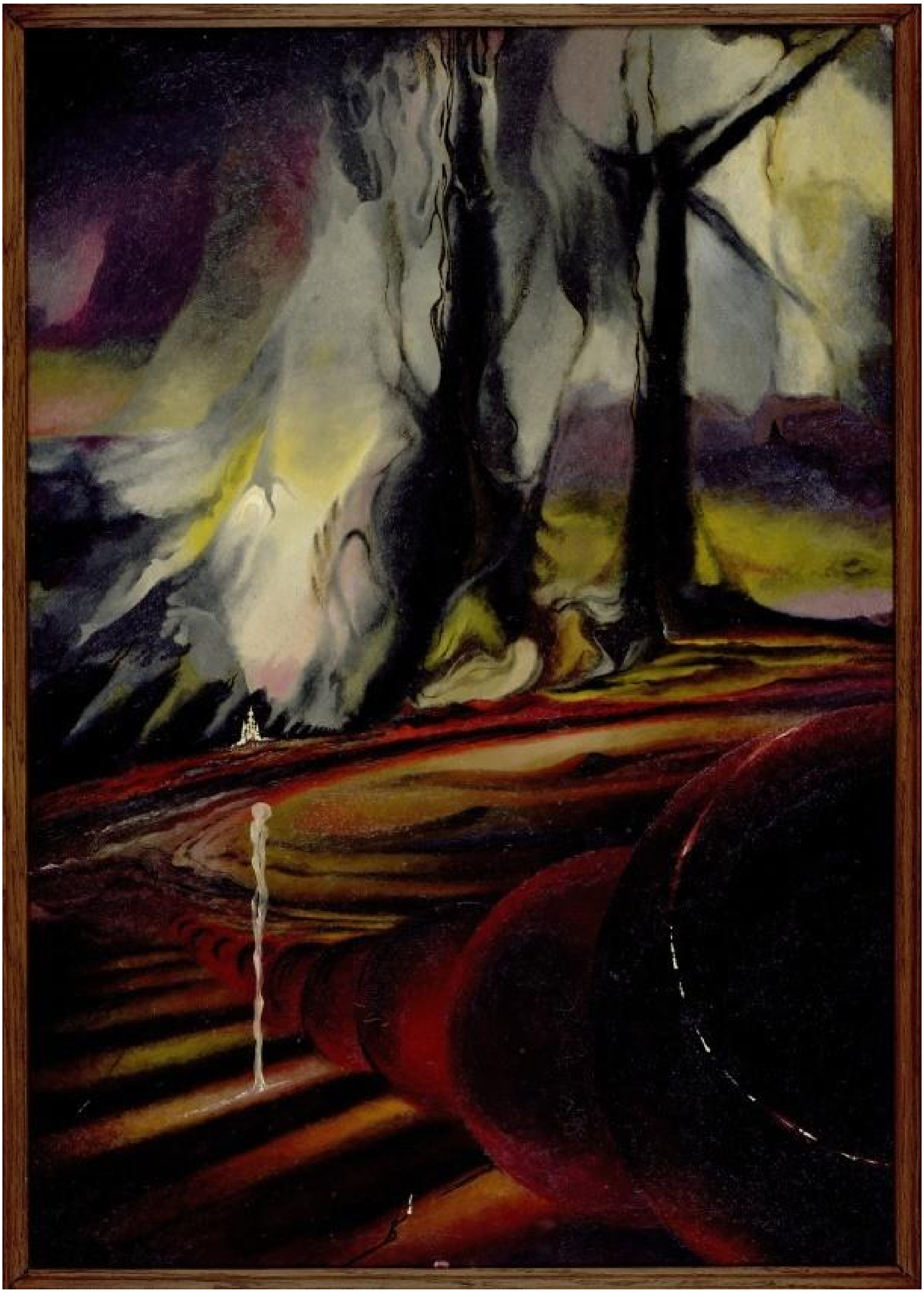
8.3 C.

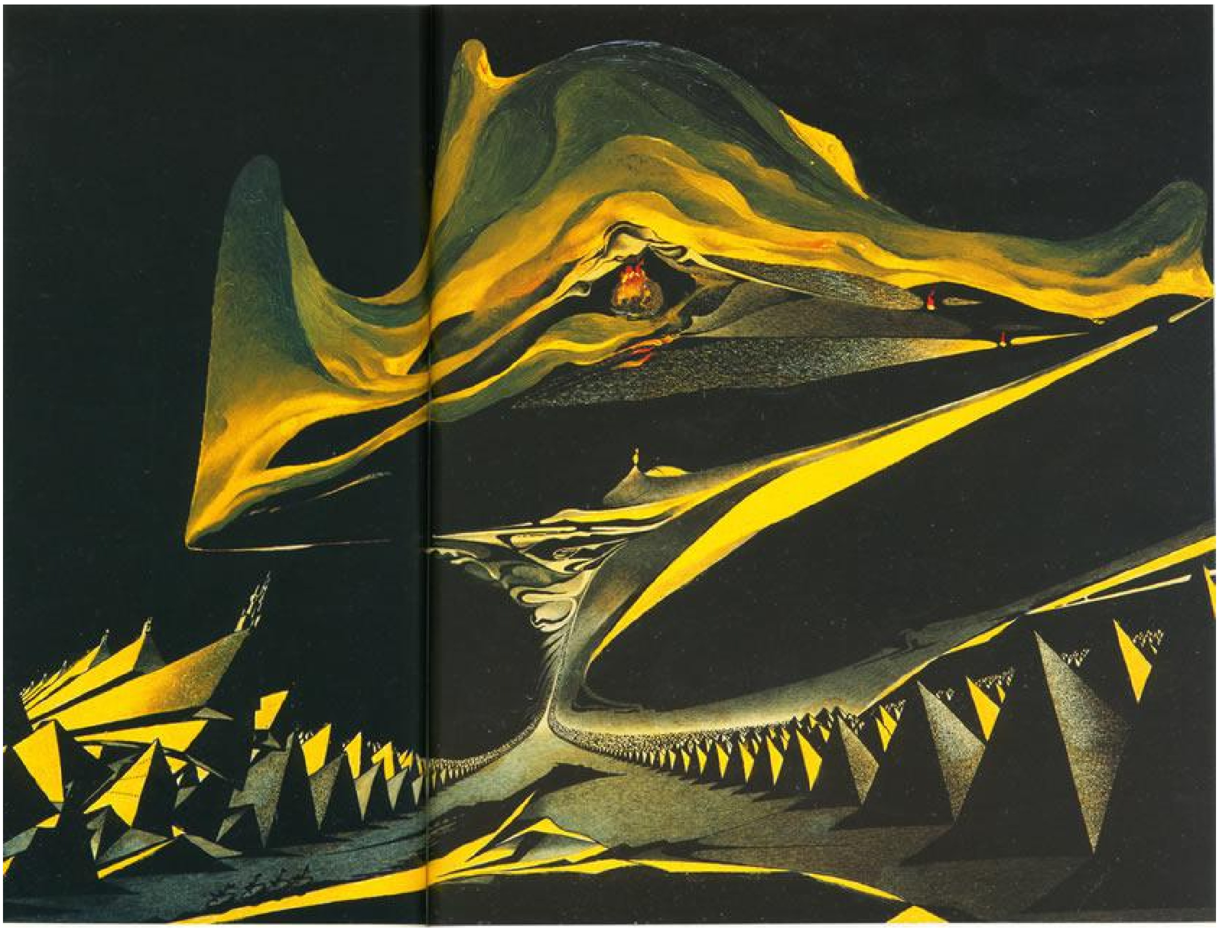
Nine artworks of Anatoly Fomenko.

1. ***A HEAVY TOP DRIFTING IN SPACE***

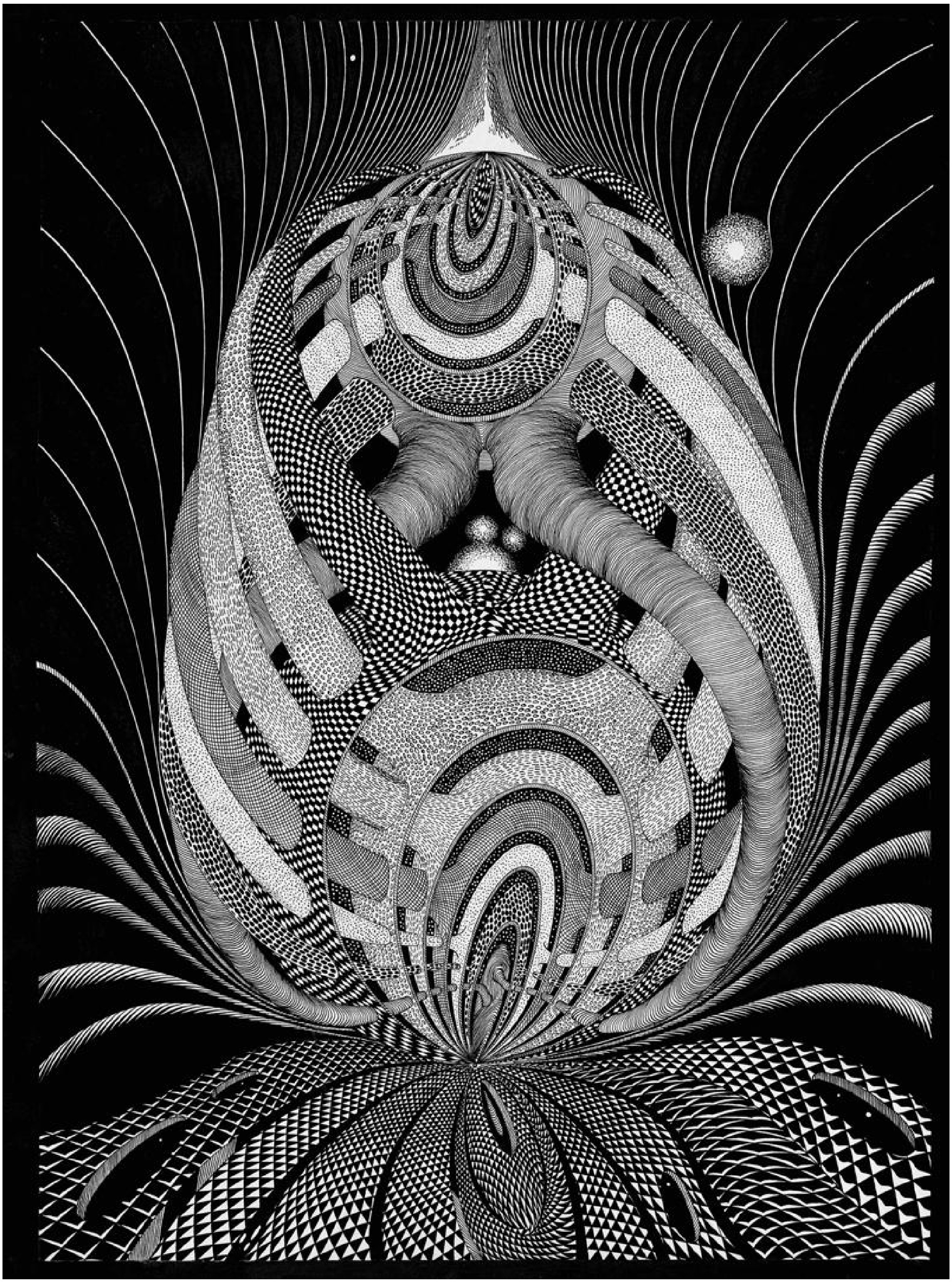


2. ***A SYSTEM OF SHRINKING NEIGHBORHOODS***

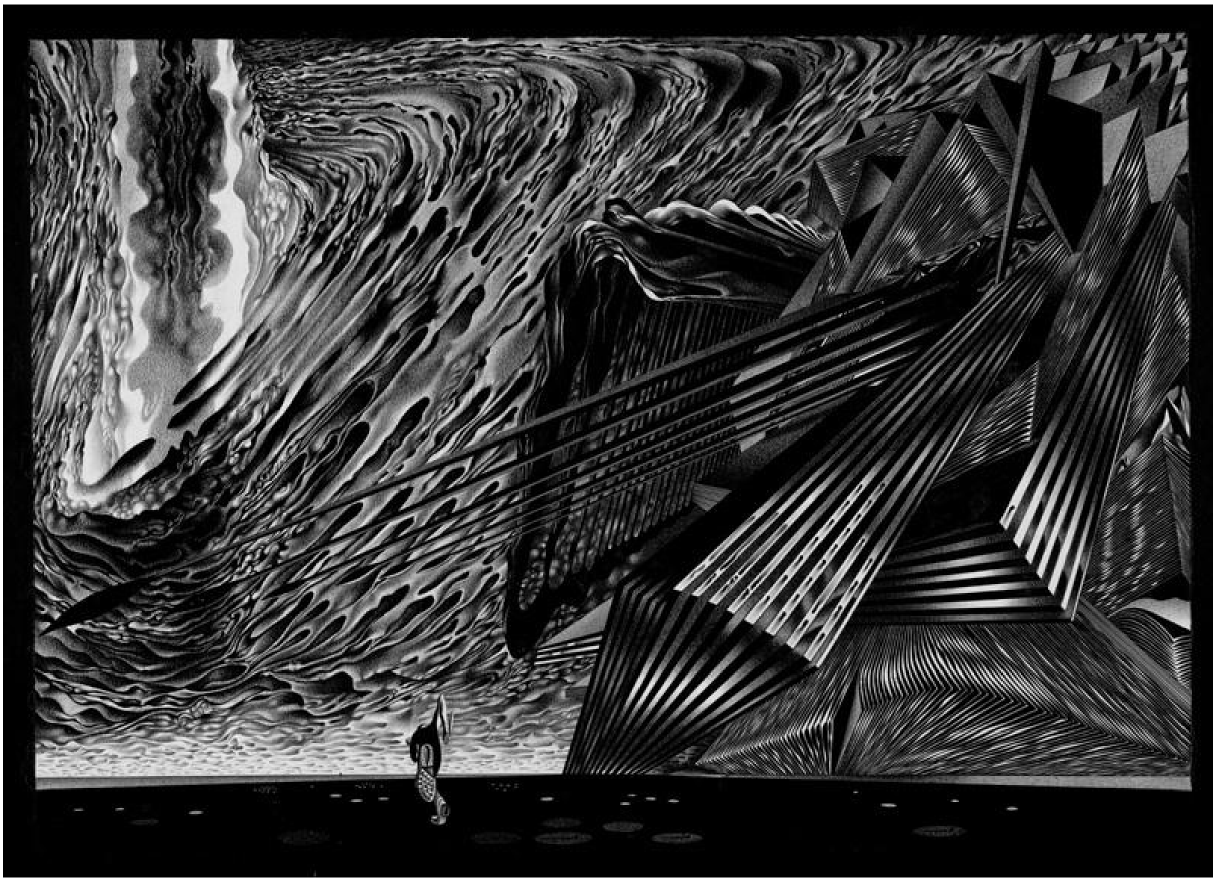




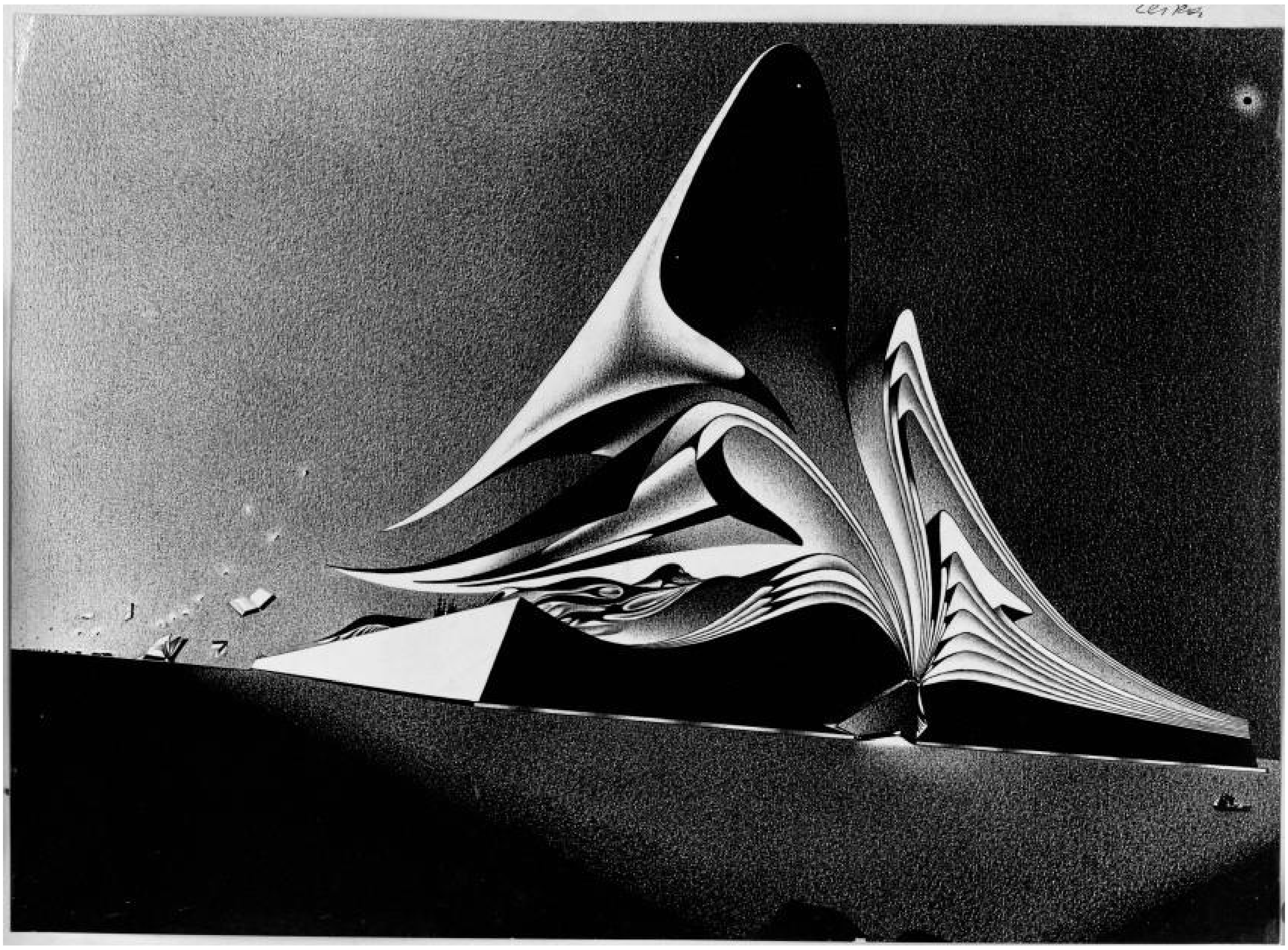
5. ***DEFORMATION OF THE RIEMANN SURFACE OF AN ALGEBRAIC FUNCTION***



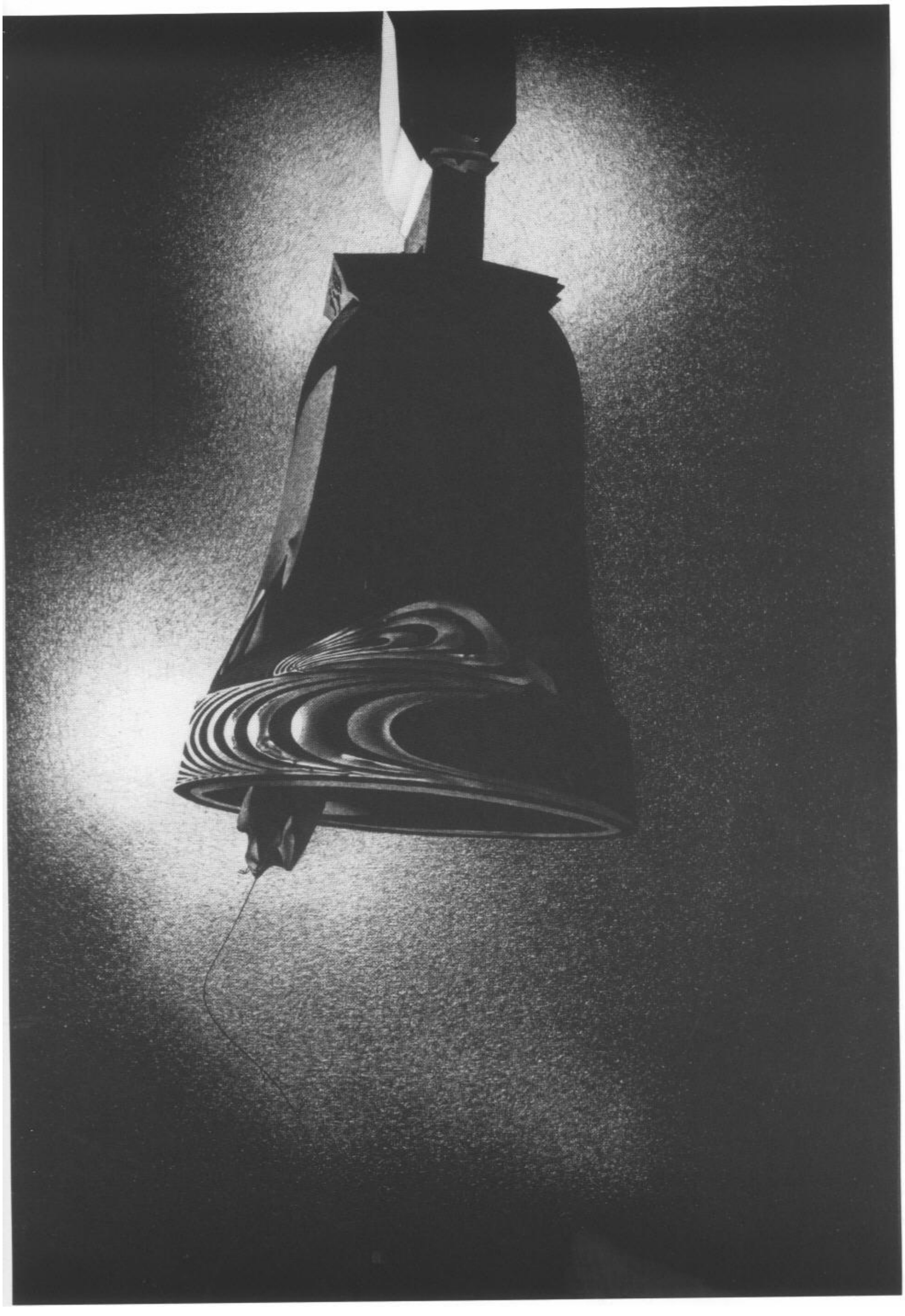
6. *SIMPLICIAL SPACES, CELLULAR SPACES, CRYSTAL AND LIQUID*

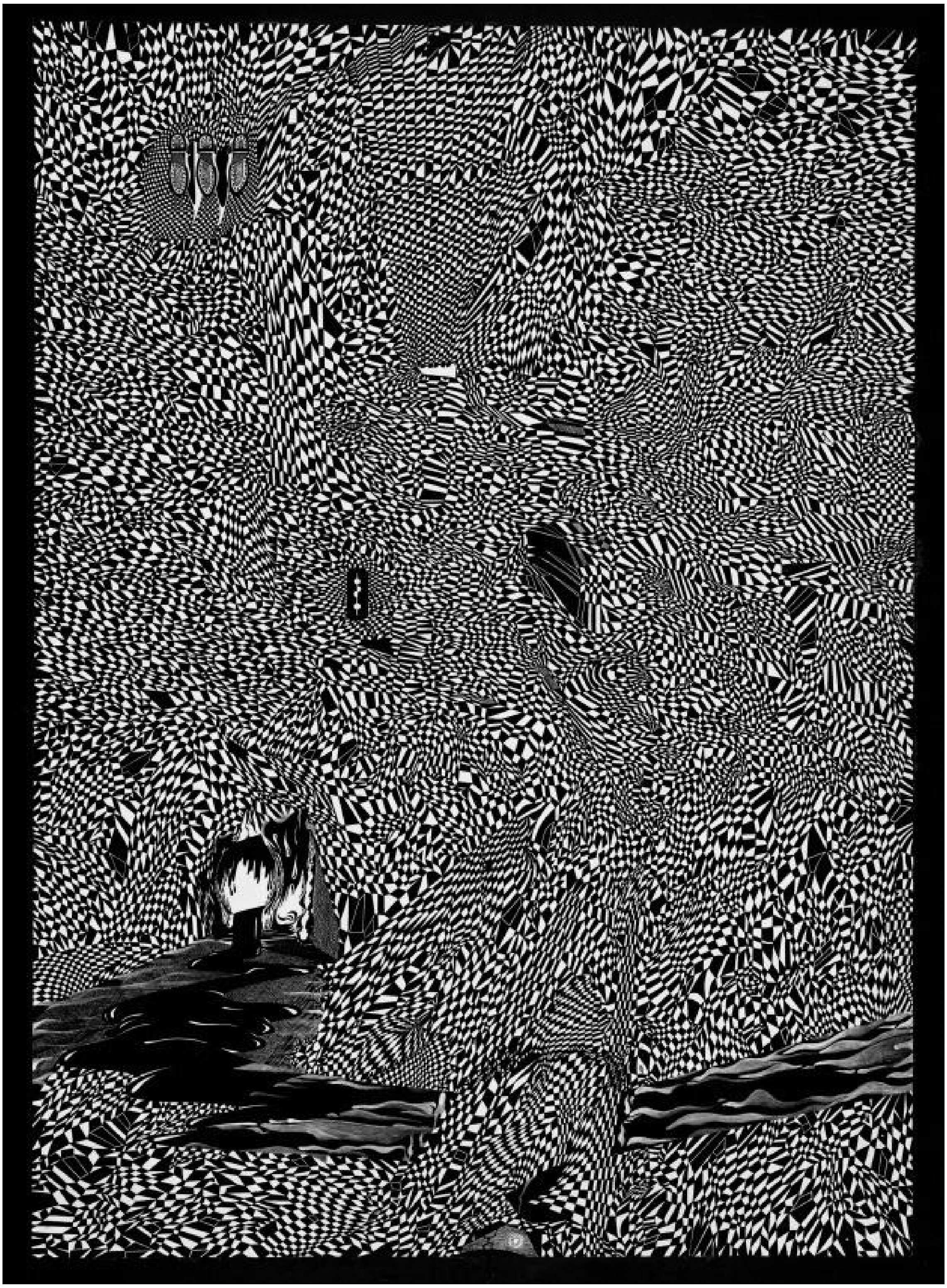


7. *SINGULARITIES OF SMOOTH FUNCTIONS*



8. *THE SEPARATIX DIAGRAM OF A CRITICAL SADDLE POINT OF A SMOOTH FUNCTION ON A 3-DIMENSIONAL MANIFOLD*

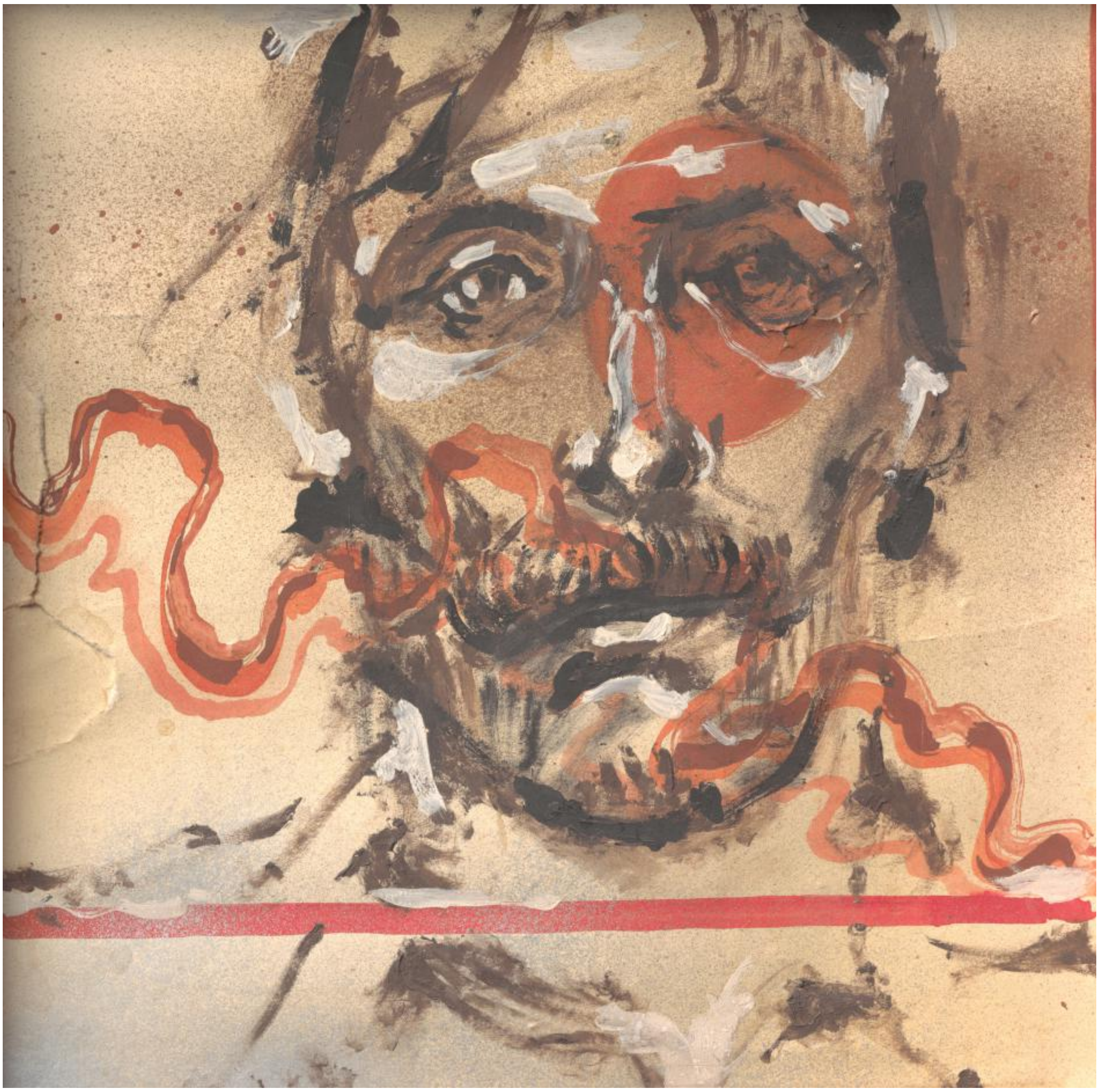




8.4 D.

Miscellaneous from the Faculty.

1. ***PROFESSOR OF LOGIC***



2. ***NOT EVEN THE MATHEMATICS CAN BE SURE***

3. ***MONSTERY LOVE***

4. ***CARD WONDERLAND***

5. ***RAINBOW GIRL***