Slovak Academy of Sciences
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Algebraic structures motivated by multiple-valued logic

Internship report

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Bratislava, 2022
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1 Multiple-Value Logic

First of all, when investigating the algebraic structures as interesting as MV-algebras, which stand for a concept of **multiple-valued** algebras, we have to get the most possible familiar with a very possibility of multiple-valued logic.

The basal difference between classical logic and multiple-valued logic, which is regarded as non-classical, is the fact that multiple-valued logic do not restrict the number of truth values to only two, but they allow a larger set of W truth degrees [1]. An example of such logic is *Łukasiewicz logic*. Łukasiewicz logic contains systems L_m and L_∞, defined in such a way that the truth degree set is either a set $W_m = \left\{\frac{k}{m-1} \mid 0 \leq k \leq m-1\right\}$ of rational numbers less or equal to 1, respectively the whole unit interval $W_\infty = \left\{x \in \mathbb{R} \mid 0 \leq x \leq 1\right\} = [0,1]$, such that the value 1 is the only designated truth degree. The main connectives of systems L_m and L_∞ are a strong and weak conjunction respectively, & and $\wedge$, defined as

$$u \mathbin{\&} v = max\ \{0, u+v-1\},$$

$$u \wedge v = min\ \{u, v\},$$

a negation connective defined by

$$\neg u = 1 - u,$$

and implication defined by

$$u \Rightarrow v = min\ \{1, 1-u+v\}.$$

Motivated with such concept, we can move to introduce a structure called multi-valued algebra.

2 MV-algebras

Definition 2.1. *An MV-algebra [2] is an algebra* $\langle A, \oplus, \neg, 0 \rangle$ *with a binary operation* $\oplus$, *a unary operation* $\neg$ *and a constant 0 satisfying the following equations:*

MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

MV2) $x \oplus y = y \oplus x$

MV3) $x \oplus 0 = x$

MV4) $\neg\neg x = x$

MV5) $x \oplus \neg 0 = \neg 0$

MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$

Example of a non-trivial MV-algebra is interval $[0, 1] = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$ equipped with truncated addition $x \oplus y \stackrel{\text{def}}{=} min\{1, x + y\}$, and negation $\neg x \stackrel{\text{def}}{=} 1 - x$.

More so, on each MV-algebra A, traditionally denoted by its universe A, we define the constant 1 and the operations $\odot$ and $\ominus$ as follows:

$1 \stackrel{\text{def}}{=} \neg 0$,

$x \odot y \stackrel{\text{def}}{=} \neg(\neg x \oplus \neg y)$,

$x \ominus y \stackrel{\text{def}}{=} x \odot \neg y$. Then, in the algebra [0,1] hold $x \odot y = max\{0, x + y - 1\}$ (which is a definition of strong conjunction u & v in Łukasiewicz logic) and $x \ominus y = max\{0, x - y\}$.

An interesting extension of the [0,1] algebra can be an interval [0,c] containing all of the possible values of velocity (we will further get straight what could be meant by a value of velocity). As numerically defined, no inherence can be in motion with a negative velocity (conceptually, the least velocity is obtained in the calm state), which might feel quite intuitive, but what is far more absorbing, no inherence can be in motion any greater than the speed of light c.

Set an algebra $\langle [0,c], \oplus, \neg, 0\rangle$ with truncated addition $\oplus$ and negation $\neg x = 1 - x$. We will show that for such algebra hold axioms of MV-algebra.

MV1) $(x \oplus y) \oplus z = min\{x + y, c\} \oplus z = min\{min\{x + y, c\} + z, c\}$. On the other hand, $x \oplus (y \oplus z) = x \oplus min\{y + z, c\} = min\{x + min\{y + z\}, c\}$.

Now might hold two cases. First, $x+y+z < c$. If that holds, also $x+y < c$ and $y+z < c$. Then, $min\{x+y, c\} \oplus z = (x+y) \oplus z = min\{x+y+z, c\} = x+y+z = x \oplus (y+z) = x \oplus min\{y+z, c\}$ and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

If $x + y + z > c$, then aside from relations of $(x + y)$, $(y + z)$ respectively, to c, hold, without loss of generality, $(x \oplus y) \oplus z = min\{x + y, c\} \oplus z = (x + y \vee c) \oplus z = min\{x + y + z, c\} \vee min\{c + z, c\} = c$, and $x \oplus (y \oplus z) = x \oplus min\{y + z, c\} = x \oplus (y + z \vee c) = min\{x + y + z, c\} \vee min\{x + c, c\} = c$, so expressly $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

Consecuting verification of *MV2) - MV5)* is trivial.

MV2) As hold $x + y = y + x$, also $x \oplus y = y \oplus x$.

MV3) $x \oplus 0 = min\{x, c\} = x$, as $x \leq c$.

MV4) $\neg\neg x = \neg(c - x) = c - (c - x) = x$.

MV5) $x \oplus \neg 0 = min\{x + \neg 0, c\} = min\{x + (c - 0), c\} = c = c - 0 = \neg 0$.

MV6) $\neg(\neg x \oplus y) \oplus y = \neg((c - x) \oplus y) \oplus y = \neg(\min\{c, c - x + y\}) \oplus y = (c - \min\{c, c - x + y\}) \oplus y = \min\{c, c - \min\{c, c - x + y\} + y\}$. The other way round, $\neg(\neg y \oplus x) \oplus x = \neg((c - y) \oplus x) \oplus x = \neg(\min\{c, c - y + x\}) \oplus x = (c - min\{c, c - y + x\}) \oplus x = \min\{c, c - \min\{c, c - y + x\} + x\}$.

Now, without loss of generality, assume that $x > y$. Hold $\min\{c, c-x+y\} = c-x+y$, so $\neg(\neg x \oplus y) \oplus y = \min\{c, c-(c-x+y)+y\} = \min\{c, x\} = x$. Similarly, $\neg(\neg y \oplus x) \oplus x = \min\{c, c-c+x\} = x$, and $\neg(\neg y \oplus x) \oplus x = \neg(\neg x \oplus y) \oplus y$. For $y > x$, comparison will behave the same way.

3 Vectors

Showed that $[0, c]$ is an MV-algebra, we can move toward executing expressly what such MV-algebra might result in under particular interpretations.

First, we see that if $\langle [0, 1], \oplus, \neg, 0\rangle$ is an MV-algebra, then $\langle [0, \mathcal{K}], \oplus, \neg, 0\rangle$ will be an MV-algebra for any $\mathcal{K} \in \mathbb{R}_0^+$. Of such side, we are having a play with an utterly abstract structure, that is possessing a potential of being found someplace in nature as a symmetry, which is, in the means of synchronicity, perfectly functional. In the subjective interpretation of an observer of such structure, an artist would indomitably stand for the most candid apprehension of a term; *beautiful*....

Second, a little less abstractly, we might consider values of velocity of the interval $[0, c]$, as has been mentioned beforehand. We know that velocity at time t_0 is a limit $\lim_{t\to t_0} \frac{s(t)-s(t_0)}{t-t_0}$ with time-dependent distance $s(t)$, so for particular considerations of *values of velocity,* we need some inherent point to be in motion under certain circumstances. What is to stun the intuition, we can do this also the other way round. We arbitrarily take values of the interval $[0, c]$ and name them values of velocity, with a maximum just abstracted to some constant $\mathcal{K}$, knowing that it is exactly $c = 299,792,458\ m.s^{-1}$, but there is still an option of scaling the interval. What should be mentioned, operations of $\oplus$ and $\neg$ remain unchanged; $[0, \mathcal{K}]$ is an MV-algebra. The amazing thing now is, velocities are *not* numbers, but as we perform operations specific for MV-algebras in general on elements of algebra $[0, \mathcal{K}]$, it will hold for velocities corresponding to these elements.

Finally, we might arrive on something intuitively resembling to a *threshold of abstraction and specificity* and have a look at the mathematical model of the behaviour of velocity in physical world. We know that inherent points in motion do move in a specific direction, with a measured value of velocity given by the mentioned limit. Therefore, we might understand the velocity development over time as a **continuous set of vectors *v(t)***, specified below.

Theorem 3.1. *In angular coordinates, vector **v(t)** might be defined as a collection of its size* s *and angles* $\theta^1, \theta^2, \ldots, \theta^n$ *corresponding to respective coordinates, such that* $0 \leq s \leq c$.

Proof. Taken in $\mathbb{R}^n$, a vector ***v(t)*** can be written as a set of coordinates $(v_1(t), v_2(t), \ldots, v_n(t))$. Moving to angular coordinates, we know ***v(t)*** is specific for its measure and direction. Therefore, we can snatch onto $s = \sqrt[k]{\sum_{i=1}^{n} v_i^k(t)}$ and realise that $v_i = f_i(s, \theta^1, \theta^2, \ldots, \theta^n)$ for all $i = 1, \ldots, n$, holding that θ^i is an angle determining position of ***v(t)*** in respect to the i-th coordinate. Then, we can imply that statements $(v_1(t), v_2(t), \ldots, v_n(t))$ and $(s, \theta^1, \theta^2, \ldots, \theta^n)$ are possessing same amount of information about ***v(t),*** and we will further confuse notations of ***v(t)***$=(v_1(t), v_2(t), \ldots, v_n(t))$ and ***v(t)***$=(s, \theta^1, \theta^2, \ldots, \theta^n)$. □

Theorem 3.2. *For* $\theta^1 = \theta^2 = \cdots = \theta^n$*, the algebra* $\langle \{\boldsymbol{v(t)}; 0 \leq s \leq c\}, \oplus, \neg, 0 \rangle$ *with* $\oplus$ *defined as*

$$\boldsymbol{v}_i(t) \oplus \boldsymbol{v}_j(t) = (s_i, \theta_i^1, \theta_i^2, \ldots, \theta_i^n) \oplus (s_j, \theta_j^1, \theta_j^2, \ldots, \theta_j^n) = (min\{s_i + s_j, c\}, \theta_i^1, \theta_i^2, \ldots, \theta_i^n) = (\min\{s_i + s_j, c\}, \theta_j^1, \theta_j^2, \ldots, \theta_j^n) = (min\{s_i + s_j, c\}, \theta^1, \theta^2, \ldots, \theta^n),$$

and

$$\neg \boldsymbol{v}_i(t) = \neg(s_i, \theta_i^1, \theta_i^2, \ldots, \theta_i^n) = (c - s_i, \theta_i^1, \theta_i^2, \ldots, \theta_i^n)$$

is an MV-algebra.

Proof. Considering the coordinate angles $\theta^1, \theta^2, \ldots, \theta^n$ remain unchanged in both the binary operation $\oplus$ and unitary operation $\neg$, and operation impact on the first coordinate is identical with operations on MV-algebra $[0, c]$, proving that $\langle [0, c], \oplus, \neg, 0\rangle$ is an MV-algebra implies that such defined $\langle \{\boldsymbol{v(t)}; 0 \leq s \leq c\}, \oplus, \neg, 0\rangle$ is an MV-algebra. □

Corollary 3.2.1. *Likewise directed velocity vectors (with corresponding binary and unitary operations defined) do behave like MV-algebra. Therefore, as inquiring after properties of a general MV-algebra, we receive an information about velocities transformed into vectors, that can potentially lead to interesting statements about velocity as such.*

Now consider two vectors differing in exactly one pair of angle coordinates, with all of the other angle coordinates equal to zero, that is represented by such assumption:

Consider two vectors $\boldsymbol{v}_k\boldsymbol{(t)}$ and $\boldsymbol{v}_\ell\boldsymbol{(t)}$ such that there is exactly one coordinate index $\boldsymbol{I}$ such that

$$(\theta_k^m, \theta_\ell^m) = \begin{cases} \mathbf{0}, & \text{if } m \neq \boldsymbol{I}, \\ (\theta_k^{\boldsymbol{I}}, \theta_\ell^{\boldsymbol{I}}), & \text{if } m = \boldsymbol{I}, \end{cases}$$

satisfying $\theta_k^{\boldsymbol{I}} \neq \theta_\ell^{\boldsymbol{I}} > 0$.

For example, if $\boldsymbol{I} \equiv 1$, hold

$$\boldsymbol{v}_k\boldsymbol{(t)} = (s_k, \theta_k^1, 0, 0, \ldots, 0),\ \boldsymbol{v}_\ell\boldsymbol{(t)} = (s_\ell, \theta_\ell^1, 0, 0, \ldots, 0),$$

satisfying $\theta_k^1 \neq \theta\ell^1 > 0$.

Now we can investigate character of addition of such velocities, represented by velocity angles.

Theorem 3.3. *Let $\boldsymbol{v}_k(t)$ and $\boldsymbol{v}_\ell(t)$ be a pair of vectors satisfying above assumption. Then, the vector addition, denote* $\boldsymbol{+}$, *will behave in a following way:*

$$\boldsymbol{v}_k(t) \boldsymbol{+} \boldsymbol{v}_\ell(t) \stackrel{\text{def}}{=} \boldsymbol{v}_{k\ell}(t),$$

and

$$\begin{aligned} s_{k\ell} &= \left(s_k^2 + s_\ell^2 + 2 s_k s_\ell (\cos(\theta_k^I)\cos(\theta_\ell^I) + \sin(\theta_k^I)\sin(\theta_\ell^I))\right)^{\frac{1}{2}}, \\ \theta_{k\ell}^I &= \arctan\left(\frac{s_k \sin\theta_k^I + s_\ell \sin\theta_\ell^I}{s_k \cos\theta_k^I + s_\ell \cos\theta_\ell^I}\right), \\ \theta_{k\ell}^i &= 0 \qquad \forall i \neq I. \end{aligned}$$

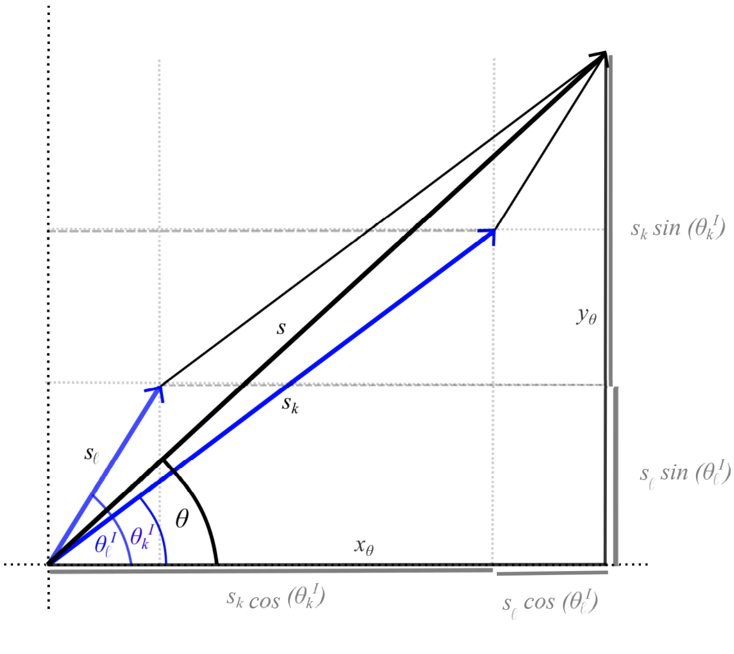

Figure 1: Polar coordinates of vector addition ($s_{k\ell} := s, \theta_{k\ell}^I := \theta$)

Proof. Issuing from *Theorem 2.1*, vector $\boldsymbol{v}_{k\ell}(t)$ will be as represented by a size $s_{k\ell}$ and corresponding angles in such a notation

$$\boldsymbol{v}_{k\ell}(t) = (s_{k\ell}, \theta_{k\ell}^1, \theta_{k\ell}^2, \dots, \theta_{k\ell}^{I-1}, \theta_{k\ell}^I, \theta_{k\ell}^{I+1}, \dots, \theta_{k\ell}^{n-1}, \theta_{k\ell}^n),$$

holding

$$(s_{k\ell}, 0, 0, \dots, 0, \theta^I_{k\ell}, 0, \dots, 0, 0).$$

Coordinates $\theta^1_{k,\ell,k\ell}, \theta^2_{k,\ell,k\ell}, \dots, \theta^{I-1}_{k,\ell,k\ell}, \theta^{I+1}_{k,\ell,k\ell}, \theta^{n-1}_{k,\ell,k\ell}, \theta^n_{k,\ell,k\ell}$ equal to zero imply that either the input vectors $\boldsymbol{v}_k\boldsymbol{(t)}$, $\boldsymbol{v}_\ell\boldsymbol{(t)}$ and the product vector of $\boldsymbol{+}, \boldsymbol{v}_{k\ell}\boldsymbol{(t)}$, lie in $\mathbb{R}^2$ determined by $\boldsymbol{I}$-th angular coordinate. Therefore, we can reduce ourselves to the general model of vector addition in $\mathbb{R}^2$. According to the Pythagorean Theorem, $s_{k\ell} = \sqrt{x_0^2 + y_0^2}$ (see Fig.1). Knowing s_k, s_ℓ, θ^I_k and θ^I_ℓ, we pick up on symmetries of the geometrical model of vector addition and write $x_0 = s_k \sin(\theta^I_k) + s_\ell \sin(\theta^I_\ell)$. Analogously, $y_0 = s_k \cos(\theta^I_k) + s_\ell \cos(\theta^I_\ell)$. Now we calculate:

$$\begin{aligned}
\mathrm{s}_{k\ell} &= \sqrt{x_0^2 + y_0^2} = \\
&= \sqrt{(s_k \sin(\theta^I_k) + s_\ell \sin(\theta^I_\ell))^2 + (s_k \cos(\theta^I_k) + s_\ell \cos(\theta^I_\ell))^2} = \\
&= \sqrt{s_k^2 + s_\ell^2 + 2 s_k s_\ell (\sin(\theta^I_k) \sin(\theta^I_\ell) + \cos(\theta^I_k) \cos\big(\theta^I_\ell\big)},
\end{aligned}$$

as $\sin(\theta^I_k)^2 + \cos(\theta^I_k)^2 = 1 = \sin(\theta^I_\ell)^2 + \cos(\theta^I_\ell)^2$.
Furthermore, $\tan(\theta^I_{k\ell}) = \frac{y_0}{x_0}$ and $\theta^I_{k\ell} \in [o, \pi]$, so

$$\theta^I_{k\ell} = \arctan\left(\frac{s_k \sin\theta^I_k + s_\ell \sin\theta^I_\ell}{s_k \cos\theta^I_k + s_\ell \cos\theta^I_\ell}\right). \qquad \square$$

4 Conclusion

Forth, it would be wantable to determine whether there is a difference between multiple-value properties of the structure of likewise oriented vectors (see *Theorem 2.2*) and the structure of vectors differing in exactly one pair of angle coordinates, with all of the other angle coordinates equal to zero (informally, differently oriented vectors). We can see that if we define the zero vector $0 \overset{\text{def}}{=} (0, 0, 0, \ldots, 0, \theta^I_0 \equiv 0, 0, \ldots, 0, 0)$ and a negation $\neg \boldsymbol{v}_k\textbf{\textit{(t)}} \overset{\text{def}}{=} (-s_k, 0, 0, \ldots, 0, \theta^I_k, 0, \ldots, 0, 0)$, a structure $\langle\{\boldsymbol{v}\textbf{\textit{(t)}}; 0 \leq s \leq c\}, \boldsymbol{+}, \neg, 0\rangle$ is not an MV-algebra, as there won't hold *MV3)* $\boldsymbol{v}_k\textbf{\textit{(t)}} \boldsymbol{+} 0 = \boldsymbol{v}_k\textbf{\textit{(t)}}$ and *MV5)* $\boldsymbol{v}_k\textbf{\textit{(t)}} \boldsymbol{+} \neg 0 = \neg 0$. Such an observation is opening questions for us: is it possible to define $\neg^*$ and 0^* in such a way that $\langle\{\boldsymbol{v}\textbf{\textit{(t)}}; 0 \leq s \leq c\}, \boldsymbol{+}, \neg^*, 0^*\rangle$ would be an MV-algebra? If so, would this new structure still be modelling physical reality? Furthermore, how is this to be interpreted. If both the structure of likewise directed vectors $\langle\{\boldsymbol{v}\textbf{\textit{(t)}}; 0 \leq s \leq c\}, \oplus, \neg, 0\rangle$ (see *Theorem 2.2*) and the structure of vectors differing in exactly one pair of angle coordinates, with all of the other angle coordinates equal to zero $\langle\{\boldsymbol{v}\textbf{\textit{(t)}}; 0 \leq s \leq c\}, \boldsymbol{+}, \neg, 0\rangle$ are MV-algebras, does this say something about vectors, and particularly, about the speed of light, as this is what motivated us to connect multiple-value logic with vectors (truncated addition)? Same question remains open for the case $\langle\{\boldsymbol{v}\textbf{\textit{(t)}}; 0 \leq s \leq c\}, \boldsymbol{+}, \neg, 0\rangle$ wouldn't be an MV-algebra. What is in the nature necessitating a difference between likewise oriented vectors and vectors in general?

5 References

[1] Gottwald, Siegfried (2015). *Many-Valued Logic*. ***Stanford Encyclopedia of Philosophy***.
https://www.researchgate.net/publication/273319231-many-valued-logic