

# INTERSECTION VOLUME OF ORIENTED RADIUS-PERPENDICULAR VECTORS GIVEN BY ANGLE VELOCITY

Dyalee, Alexandra

july 2023

## Description of a problem.

Linking to the paper *INTERSECTION DENSITY OF ORIENTED RADIUS-PERPENDICULAR HALF-LINES*, we have a density probability function of half-line conjunctions led from given  $k \rightarrow \infty$  diagonals in distance  $d = |Od|$ . Hold

$$P(d) = \begin{cases} \frac{1}{2}, & \text{if } d \leq r, \\ \frac{\arcsin\left(\frac{r}{d}\right)}{\pi}, & \text{if } d > r. \end{cases}$$

for  $d \in \mathbb{R}^+$ .

Now, we will suppose that instead of radius-perpendicular half-lines, we have radius-perpendicular vectors given by a (vector) function  $\vec{v}(t) = \omega t$ ,  $t \in [0, r]$ ,  $\omega$  is a given angular velocity (Fig. 1).

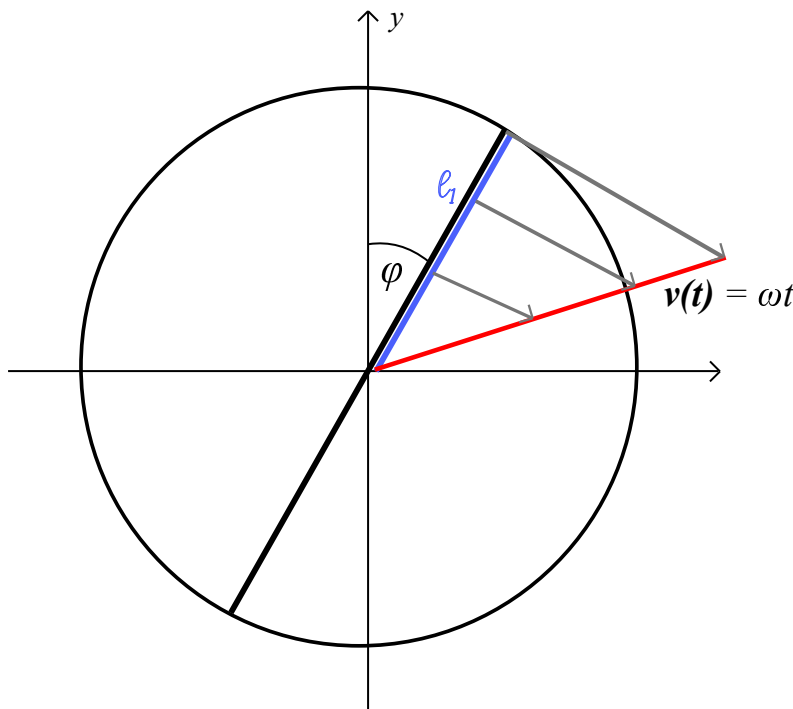


Figure 1: Radius-perpendicular vectors. Schematic figure.

Without loss of generality suppose that every vector  $\vec{v}(t)$  with a correct trajectory will intersect a point  $\mathcal{P} \in \ell_1$ ,  $|\mathcal{P}O| = d$ , exactly once.

We will be finding a function  $\Sigma(d)$  computing the **sum of velocities of vectors intersecting point  $\mathcal{P}$  on radius-perpendicular trajectory.**

# INTERSECTION VOLUME OF ORIENTED RADIUS-PERPENDICULAR VECTORS GIVEN BY ANGLE VELOCITY

Dyalee, Alexandra

july 2023

**Solution.**

Consider a point  $\mathcal{P}$  such that  $d \geq r$ , satisfying above assumptions. Taking the fact that trajectory of each vector is a line perpendicular to  $\ell_1$  (radius-perpendicular), we see that all of the points from which there can lead a vector intersecting  $\mathcal{P}$  can be (according to Thales Theorem) expressed as a part of circumference of a circle  $c_1(O_1, r_1) \equiv c_1([\frac{d}{2} \cos(\frac{\pi}{2} - \varphi), \frac{d}{2} \sin(\frac{\pi}{2} - \varphi)], \frac{d}{2})$  (see Fig. 2).

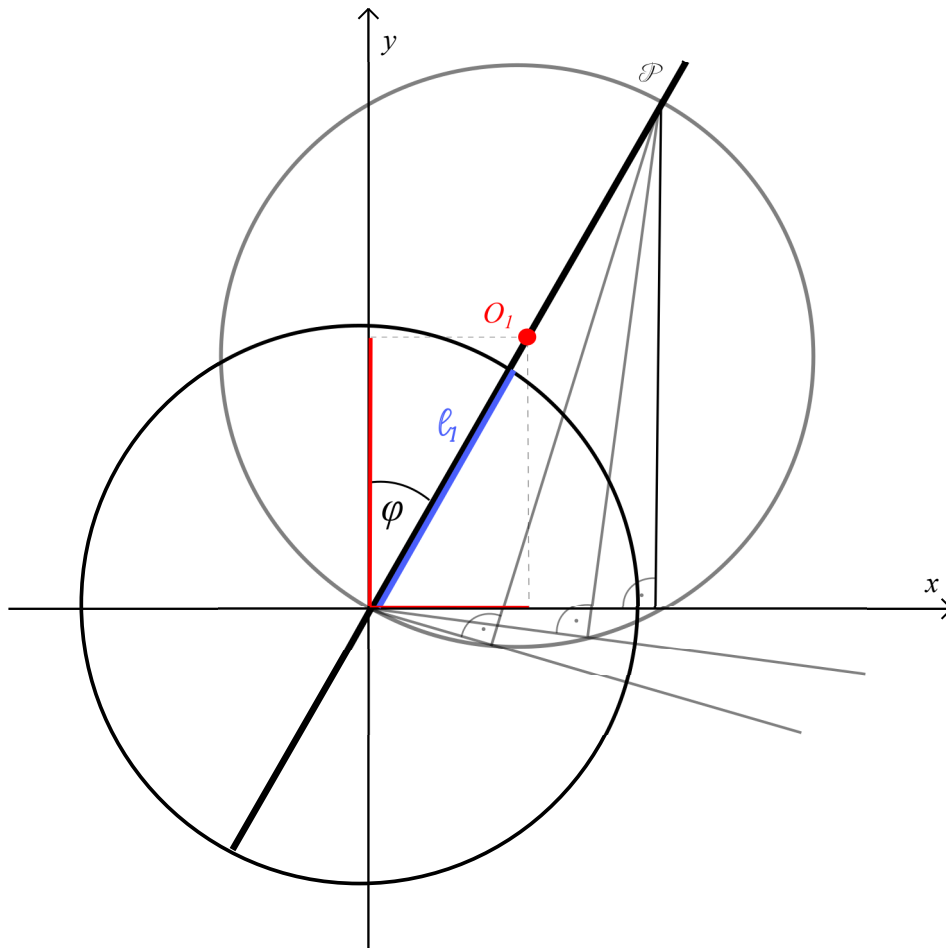


Figure 2: Vectors intersecting  $\mathcal{P}$  (1)

This expression is equal to a circle  $c_1([\frac{d}{2} \sin(\varphi), \frac{d}{2} \cos(\varphi)], \frac{d}{2})$ .

We will calculate a curve integral of the first type along such part of circle (in respect of  $d$ ) of a function  $v(t) = \omega t; t \in [0, r]$ . Note that we considered vector orientation in the assumption that all of vectors are radius-perpendicular and out of that we devised trajectory. Therefore, it will be sufficient for us to calculate the density among trajectory, which is a curve integral of the first type.

# INTERSECTION VOLUME OF ORIENTED RADIUS-PERPENDICULAR VECTORS GIVEN BY ANGLE VELOCITY

Dyalee, Alexandra

july 2023

We have to find parametrisation of the trajectory. Be denoted  $\mathcal{K}(\theta)$ . Using polar coordinates, we have  $\mathcal{K}(\theta) = \{(x_0 + \frac{d}{2} \cos(\theta), y_0 + \frac{d}{2} \sin(\theta)); \theta \in [0, \arccos(1 - \frac{r^2}{d^2})]\} = \{(\frac{d}{2} \sin(\varphi) + \frac{d}{2} \cos(\theta), \frac{d}{2} \cos(\varphi) + \frac{d}{2} \sin(\theta)); \theta \in [0, \arccos(1 - \frac{r^2}{d^2})]\}$ . The range of parameter  $\theta$  is calculated using the Law of Cosines from an isosceles triangle  $\triangle OO_1Q$  (see Fig. 3).

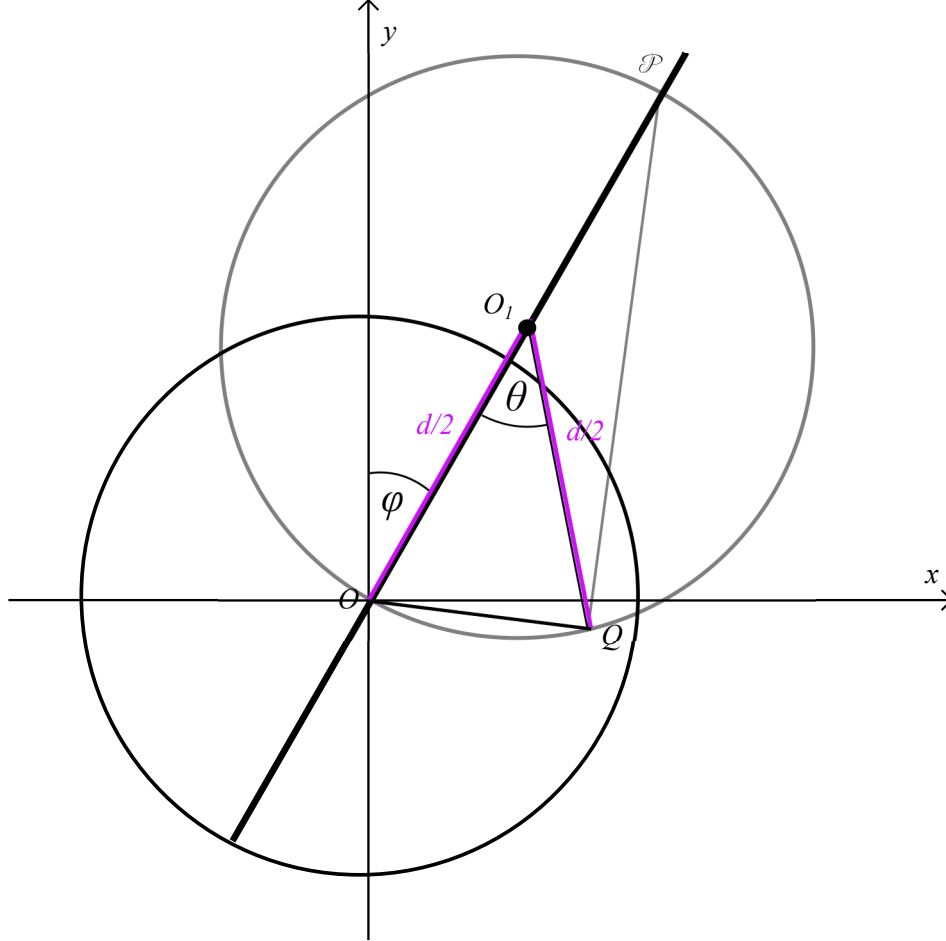


Figure 3: Range of parameter  $\theta$

Without loss of generality, suppose  $\varphi = \frac{\pi}{4}$ . Then finally  $\mathcal{K}(\theta) = \{(\frac{\sqrt{2}}{4}d + \frac{d}{2} \cos(\theta), \frac{\sqrt{2}}{4}d + \frac{d}{2} \sin(\theta)); \theta \in [0, \arccos(1 - \frac{R^2}{d^2})]\}$ .

So we have  $\Sigma(d) = \int_{\mathcal{K}} v(\theta) ds$ . We have to express  $v(t) = \omega t$  in respect of angle  $\theta$ . We will continue analogously – according to the Law of Cosines, we can devise  $t = \frac{\sqrt{2}}{2}d\sqrt{1 - \cos(\theta)}$ . So hold  $v(t) = \omega t = v(\theta) = \omega f(\theta) = \frac{\sqrt{2}}{2}\omega d\sqrt{1 - \cos(\theta)}$ . According to definition, we have  $ds = \sqrt{x'^2(\theta) + y'^2(\theta)} d\theta$ . Since  $x'(\theta) = \frac{d}{2} \sin(\theta)$  and  $y'(\theta) = -\frac{d}{2} \cos(\theta)$ , hold  $ds = \sqrt{\frac{d^2}{4}} d\theta = \frac{d}{2} d\theta$ . So we are finding  $\Sigma(d) = \int_0^{\arccos(1 - \frac{r^2}{d^2})} \frac{\sqrt{2}}{2}\omega d\sqrt{1 - \cos(\theta)} \frac{d}{2} d\theta$ . Since  $\cos(\theta) = \cos^2(\frac{\theta}{2}) -$

# INTERSECTION VOLUME OF ORIENTED RADIUS-PERPENDICULAR VECTORS GIVEN BY ANGLE VELOCITY

Dyalee, Alexandra

july 2023

$\sin^2\left(\frac{\theta}{2}\right)$ ,  $\Sigma(d) = \frac{\omega d^2}{2} \int_0^{\psi(d)} \sin\left(\frac{\theta}{2}\right) d\theta$ . Substituting  $\frac{\theta}{2} = \alpha$ , we get  $\Sigma(d) = \frac{\sqrt{2}}{2} \omega d^2 \int_0^{\frac{\psi(d)}{2}} \sin(\alpha) d\alpha = \frac{\sqrt{2}}{2} \omega d^2 (1 - \cos \frac{\psi(d)}{2})$ . This is finally equal to

$$\Sigma(d) = \frac{\sqrt{2}}{2} \omega d^2 \left( 1 - \cos \left( \frac{\arccos \left( 1 - \frac{r^2}{d^2} \right)}{2} \right) \right); d \geq r.$$

In the case of  $d < r$ , note that angle  $\theta$  is always equal to  $\frac{\pi}{2}$  (we can devise it from the Law of Sines, analogously as in the first case from isosceles triangle  $\triangle OO_1Q$  – see Fig. 4).

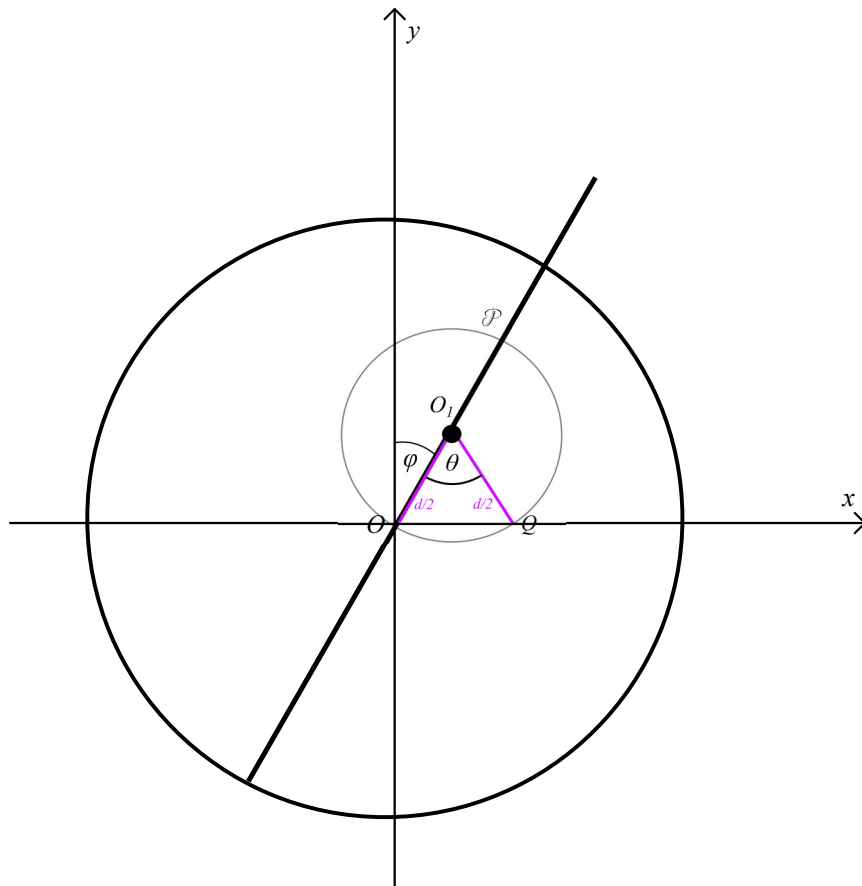


Figure 4: Parameter  $\theta$  in regard of  $d < r$  (schematic figure)

Therefore, we have

$$\Sigma(d) = \frac{\sqrt{2}}{2} \omega d^2 \left( 1 - \cos \left( \frac{\frac{\pi}{2}}{2} \right) \right) = \left( \sqrt{2} - \frac{1}{2} \right) \omega d^2; d < r.$$

# INTERSECTION VOLUME OF ORIENTED RADIUS-PERPENDICULAR VECTORS GIVEN BY ANGLE VELOCITY

Dyalee, Alexandra

july 2023

Finally, we have

$$\Sigma(d) = \begin{cases} \frac{\sqrt{2}}{2} \omega d^2 \left( 1 - \cos \left( \frac{\arccos \left( 1 - \frac{r^2}{d^2} \right)}{2} \right) \right), & \text{if } d \geq r, \\ \left( \sqrt{2} - \frac{1}{2} \right) \omega d^2, & \text{if } d < r. \end{cases}$$

To conclude, the function expresses the sum of vector velocities at a given point, depending merely on the distance  $d$ . For the physical model, we suppose we have fixed angle velocity  $\omega$  and radius  $r$ .

Fix (irrespectively of units)  $\omega \equiv 1, r \equiv 16$ . Graphically, the resulting function looks following (Fig. 5).

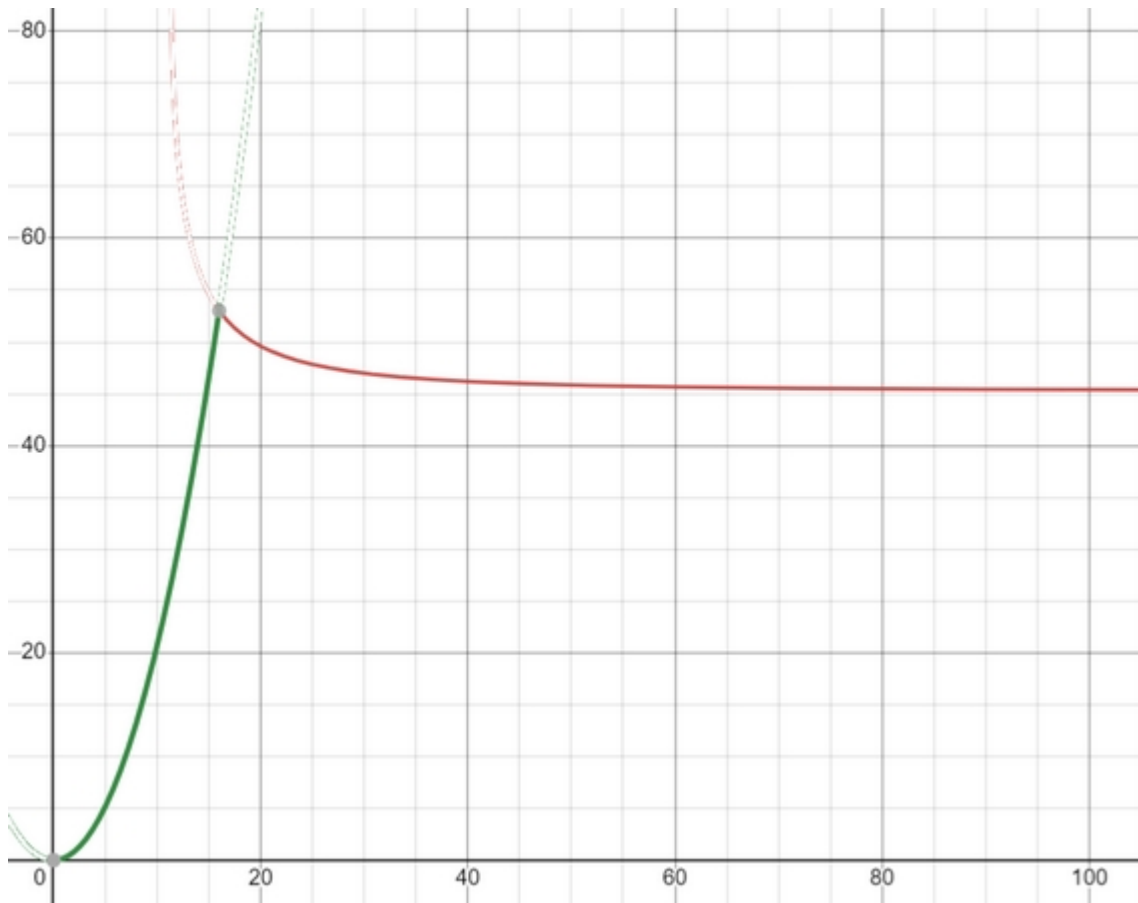


Figure 5:  $\Sigma(d)$  for  $\omega \equiv 1, r \equiv 16$

We can see that maximum of  $\Sigma(d)$  is a value  $\Sigma(r)$ , that **increases with increasing both parameters  $\omega$  and  $r$**  – see Fig. 6 with  $\omega \equiv 3, r \equiv 50$ .

# INTERSECTION VOLUME OF ORIENTED RADIUS-PERPENDICULAR VECTORS GIVEN BY ANGLE VELOCITY

Dyalee, Alexandra

july 2023

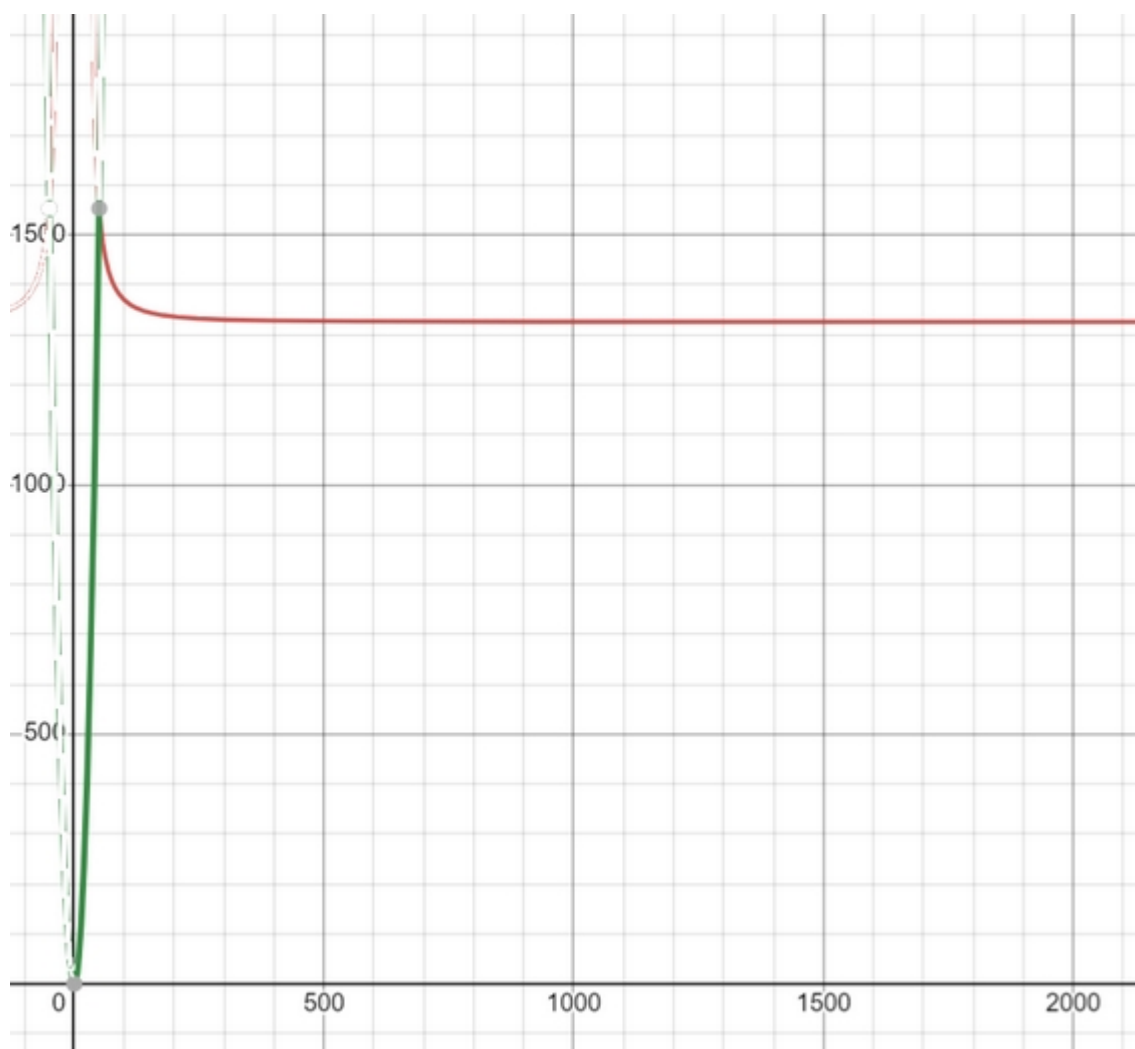


Figure 6:  $\Sigma(d)$  for  $\omega \equiv 3, r \equiv 50$