INTERSECTION VOLUME OF ORIENTED RADIUS-PERPENDICULAR VECTORS GIVEN BY ANGLE VELOCITY

Dyalee, Alexandra july 2023

Description of a problem.

Linking to the paper *INTERSECTION DENSITY OF ORIENTED RADIUS-PERPENDICULAR HALF-LINES*, we have a density probability function of half-line conjunctions led from given $k \to \infty$ diagonals in distance $d = |Od|$. Hold

$$P(d) = \begin{cases} \frac{1}{2}, & \text{if } d \leq r, \\ \frac{\arcsin\left(\frac{r}{d}\right)}{\pi}, & \text{if } d > r. \end{cases}$$

for $d \in \mathbb{R}^+$.

Now, we will suppose that instead of radius-perpendicular half-lines, we have radius-perpendicular vectors given by a (vector) function $\overrightarrow{v(t)} = \omega t$, $t \in [0, r]$, ω is a given angular velocity (Fig. 1).

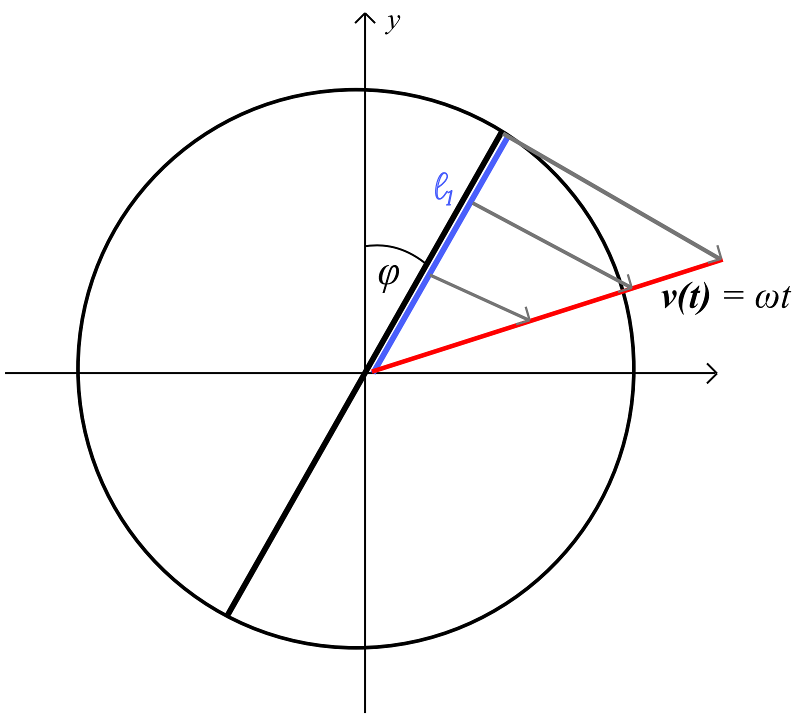

Figure 1: Radius-perpendicular vectors. Schematic figure.

Without loss of generality suppose that every vector $\overrightarrow{v(t)}$ with a correct trajectory will intersect a point $\mathcal{P} \in \ell_1, |\mathcal{P}O| = d$, exactly once.

We will be finding a function $\Sigma(d)$ computing the **sum of velocities of vectors intersecting point $\mathcal{P}$ on radius-perpendicular trajectory.**

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Solution.

Consider a point $\mathcal{P}$ such that $d \geq r$, satisfying above assumptions. Taking the fact that trajectory of each vector is a line perpendicular to ℓ_1 (radius-perpendicular), we see that all of the points from which there can lead a vector intersecting $\mathcal{P}$ can be (according to Thales Theorem) expressed as a part of circumference of a circle $c_1(O_1, r_1) \equiv c_1([\frac{d}{2}\cos(\frac{\pi}{2} - \varphi), \frac{d}{2}\sin(\frac{\pi}{2} - \varphi)], \frac{d}{2})$ (see Fig. 2).

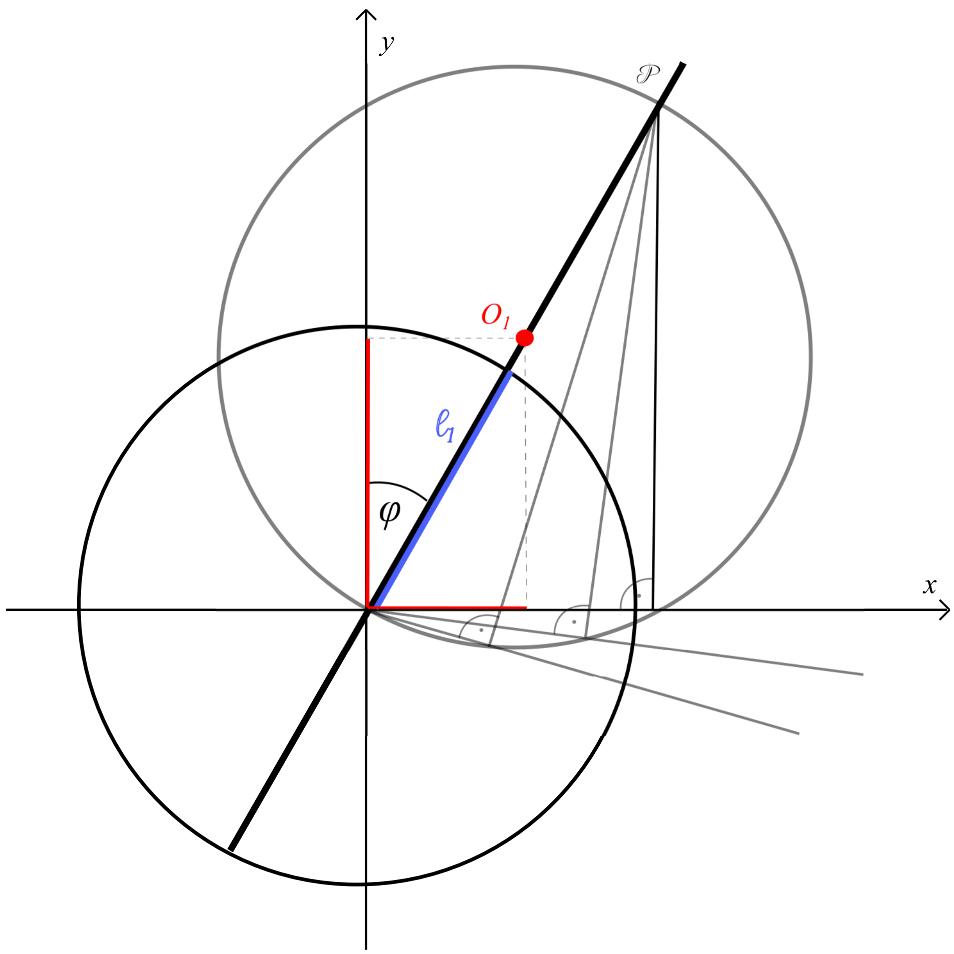

Figure 2: Vectors intersecting $\mathcal{P}$ (1)

This expression is equal to a circle $c_1([\frac{d}{2}\sin(\varphi), \frac{d}{2}\cos(\varphi)], \frac{d}{2})$.

We will calculate a curve integral of the first type along such part of circle (in respect of d) of a function $v(t) = \omega t; t \in [0, r]$. Note that we considered vector orientation in the assumption that all of vectors are radius-perpendicular and out of that we devised trajectory. Therefore, it will be sufficient for us to calculate the density among trajectory, which is a curve integral of the first type.

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We have to find parametrisation of the trajectory. Be denoted $\mathcal{K}(\theta)$. Using polar coordinates, we have $\mathcal{K}(\theta) = \{(x_0 + \frac{d}{2}\cos(\theta), y_0 + \frac{d}{2}\sin(\theta)); \theta \in [0, \arccos(1 - \frac{r^2}{d^2})]\} = \{(\frac{d}{2}\sin(\varphi) + \frac{d}{2}\cos(\theta), \frac{d}{2}\cos(\varphi) + \frac{d}{2}\sin(\theta)); \theta \in [0, \arccos(1 - \frac{r^2}{d^2})]\}$. The range of parameter θ is calculated using the Law of Cosines from an isosceles triangle $\triangle OO_1Q$ (see Fig. 3).

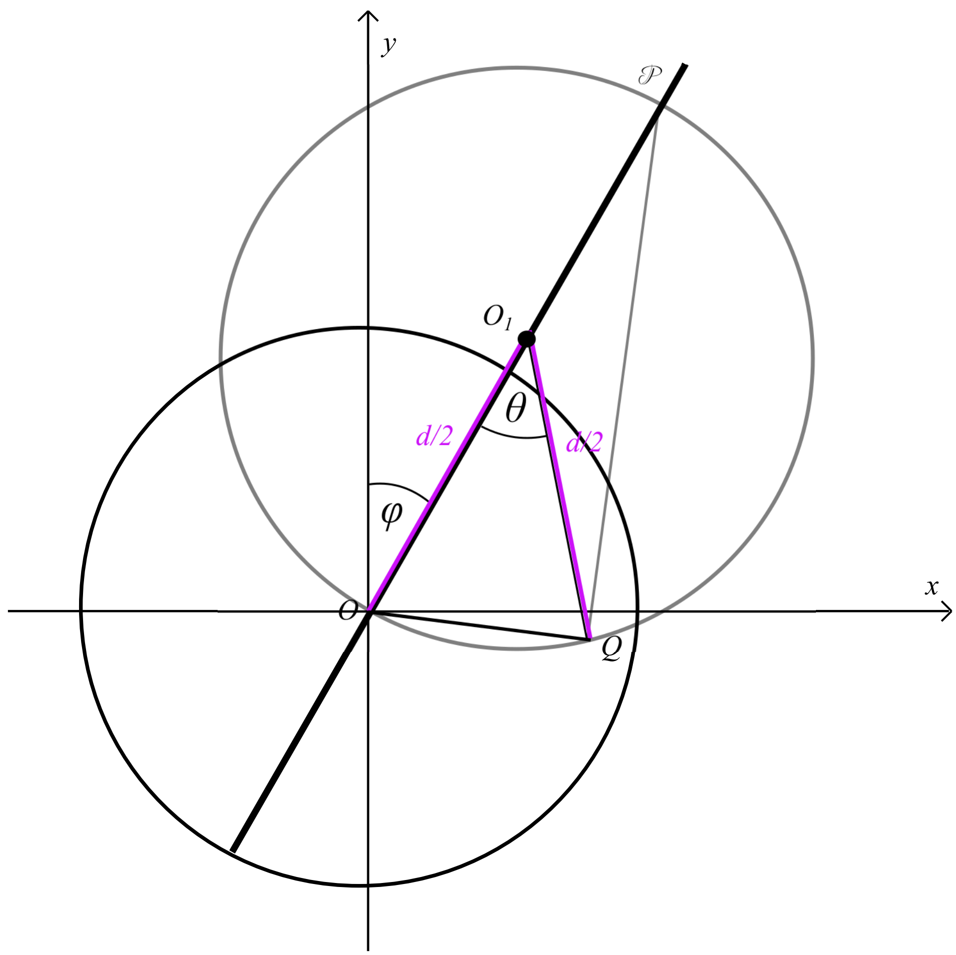

Figure 3: Range of parameter θ

Without loss of generality, suppose $\varphi = \frac{\pi}{4}$. Then finally $\mathcal{K}(\theta) = \{(\frac{\sqrt{2}}{4}d + \frac{d}{2}\cos(\theta), \frac{\sqrt{2}}{4}d + \frac{d}{2}\sin(\theta)); \theta \in [0, \arccos(1 - \frac{R^2}{d^2})]\}$.

So we have $\Sigma(d) = \int_{\mathcal{K}} v(\theta)\ \mathrm{ds}$. We have to express $v(t) = \omega t$ in respect of angle θ. We will continue analogously – according to the Law of Cosines, we can devise $t = \frac{\sqrt{2}}{2}d\sqrt{1-\cos(\theta)}$. So hold $v(t) = \omega t = v(\theta) = \omega f(\theta) = \frac{\sqrt{2}}{2}\omega d\sqrt{1-\cos(\theta)}$. According to definition, we have ds $= \sqrt{x'^2(\theta) + y'^2(\theta)}\ \mathrm{d}\theta$. Since $x'(\theta) = \frac{d}{2}\sin(\theta)$ and $y'(\theta) = -\frac{d}{2}\cos(\theta)$, hold ds $= \sqrt{\frac{d^2}{4}}\ \mathrm{d}\theta = \frac{d}{2}$ $\mathrm{d}\theta$. So we are finding $\Sigma(d) = \int_0^{\arccos\left(1-\frac{r^2}{d^2}\right)} \frac{\sqrt{2}}{2}\omega d\sqrt{1-\cos(\theta)}\frac{d}{2}\ \mathrm{d}\theta$. Since $\cos(\theta) = \cos^2\left(\frac{\theta}{2}\right) -$

$\sin^2\left(\frac{\theta}{2}\right)$, $\Sigma(d) = \frac{\omega d^2}{2}\int_0^{\psi(d)} \sin\left(\frac{\theta}{2}\right)\,\mathrm{d}\theta$. Substituting $\frac{\theta}{2} = \alpha$, we get $\Sigma(d) = \frac{\sqrt{2}}{2}\omega d^2 \int_0^{\frac{\psi(d)}{2}} \sin(\alpha)\,\mathrm{d}\alpha = \frac{\sqrt{2}}{2}\omega d^2\big(1 - \cos\frac{\psi(d)}{2}\big)$. This is finally equal to

$$\Sigma(d) = \frac{\sqrt{2}}{2}\omega d^2 \left(1 - \cos\left(\frac{\arccos\left(1 - \frac{r^2}{d^2}\right)}{2}\right)\right); d \geq r.$$

In the case of $d < r$, note that angle θ is always equal to $\frac{\pi}{2}$ (we can devise it from the Law of Sines, analogously as in the first case from isosceles triangle $\triangle OO_1Q$ – see Fig. 4).

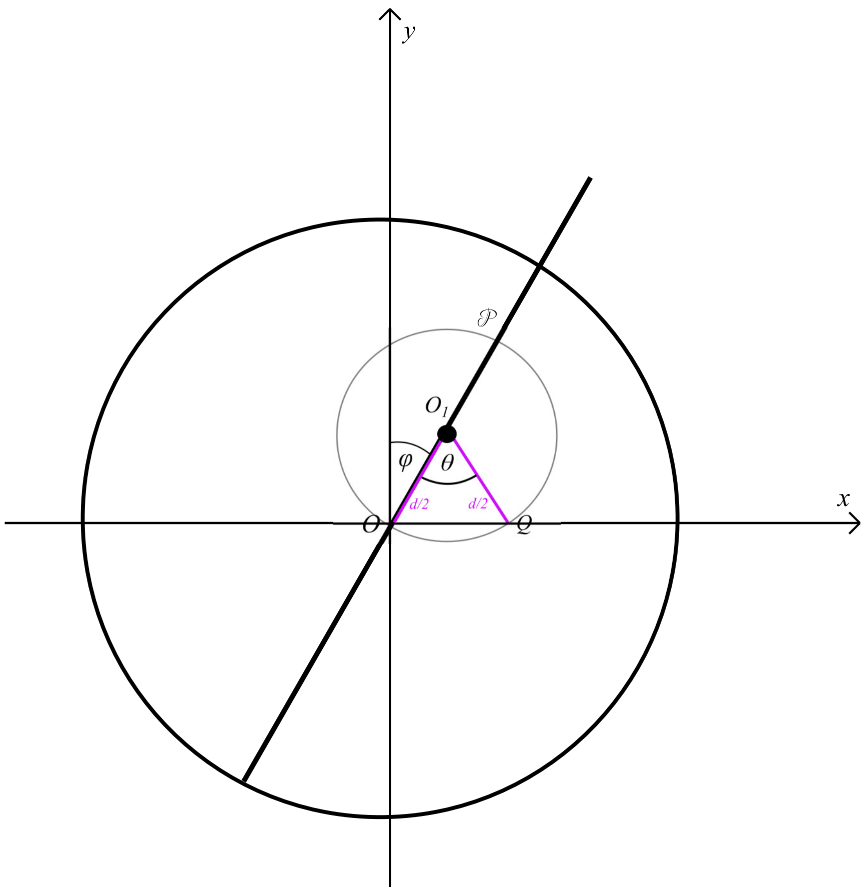

Figure 4: Parameter θ in regard of $d < r$ (schematic figure)

Therefore, we have

$$\Sigma(d) = \frac{\sqrt{2}}{2}\omega d^2 \left(1 - \cos\left(\frac{\frac{\pi}{2}}{2}\right)\right) = \left(\sqrt{2} - \frac{1}{2}\right)\omega d^2; d < r.$$

Finally, we have

$$\Sigma(d) = \begin{cases} \frac{\sqrt{2}}{2}\omega d^2 \left(1 - \cos\left(\frac{\arccos\left(1-\frac{r^2}{d^2}\right)}{2}\right)\right), & \text{if } d \geq r, \\ \left(\sqrt{2} - \frac{1}{2}\right)\omega d^2, & \text{if } d < r. \end{cases}$$

To conclude, the function expresses the sum of vector velocities at a given point, depending merely on the distance d. For the physical model, we suppose we have fixed angle velocity ω and radius r.

Fix (irrespectively of units) $\omega \equiv 1, r \equiv 16$. Graphically, the resulting function looks following (Fig. 5).

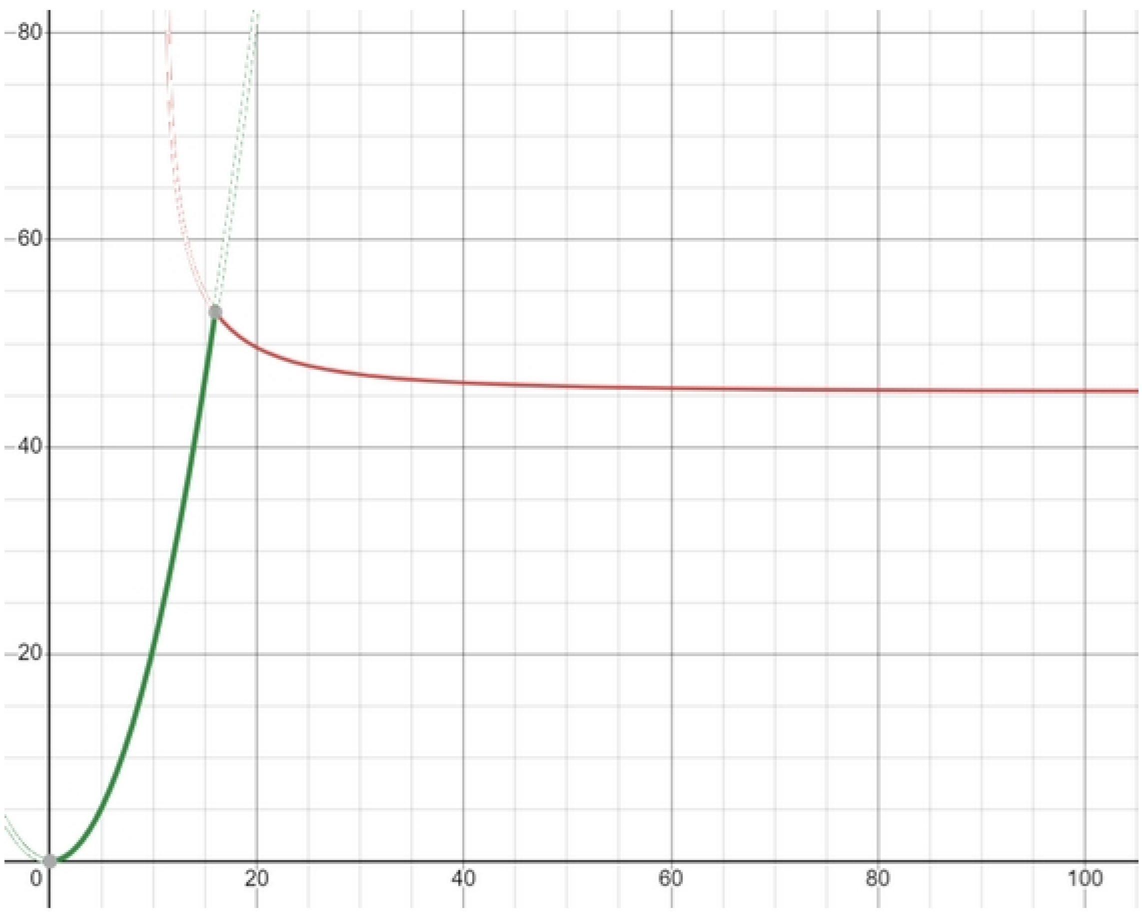

Figure 5: $\Sigma(d)$ for $\omega \equiv 1, r \equiv 16$

We can see that maximum of $\Sigma(d)$ is a value $\Sigma(r)$, that **increases with increasing both parameters ω and r** – see Fig. 6 with $\omega \equiv 3, r \equiv 50$.

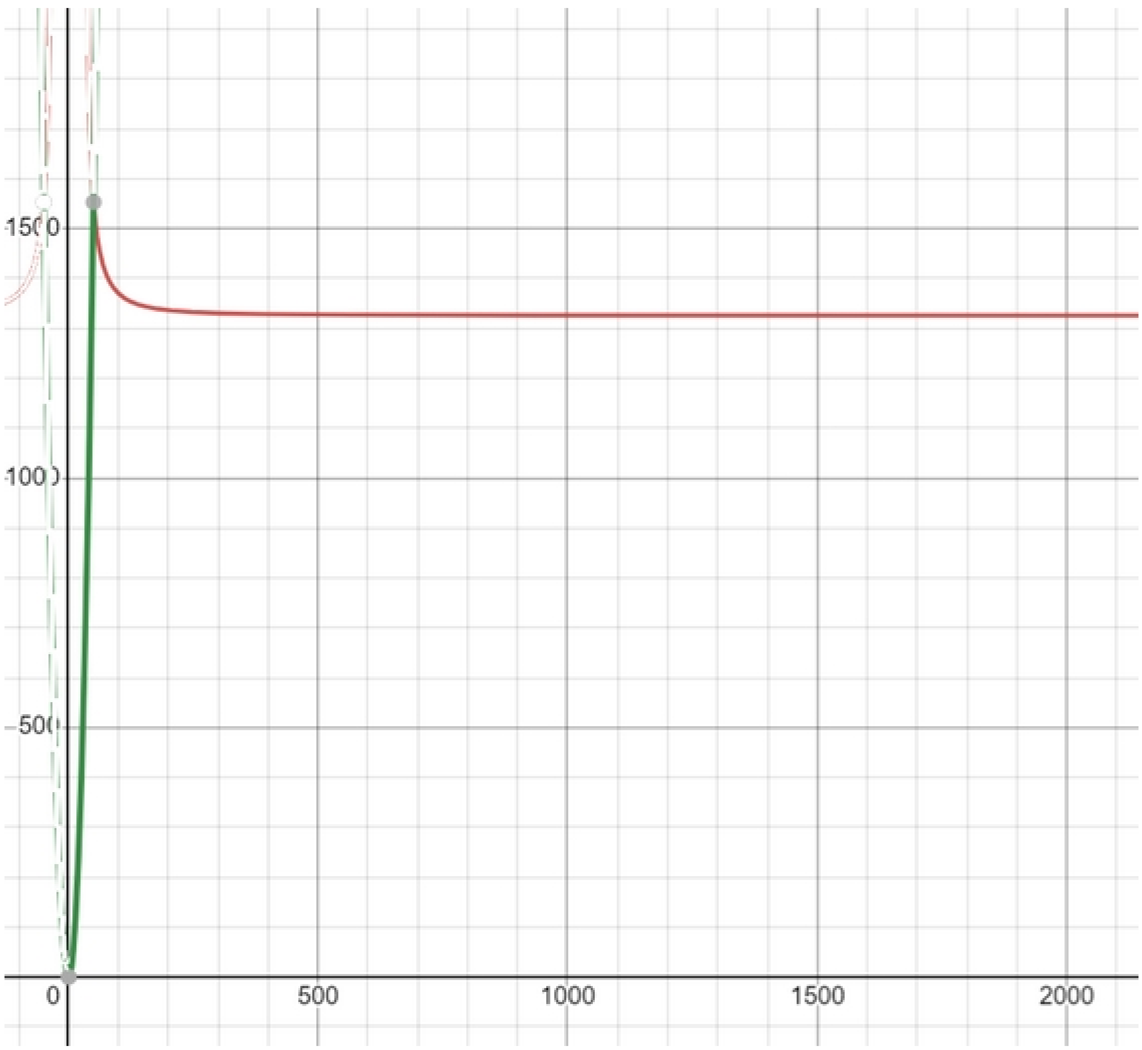

Figure 6: $\Sigma(d)$ for $\omega \equiv 3, r \equiv 50$