INTERSECTION DENSITY OF ORIENTED RADIUS-PERPENDICULAR HALF-LINES

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Description of a problem.

Consider a circle c in Cartesian's coordinate system O_{xy}, with a radius R. In this circle, consider a diagonal ℓ deflected from y-axis by angle φ (Figure 1).

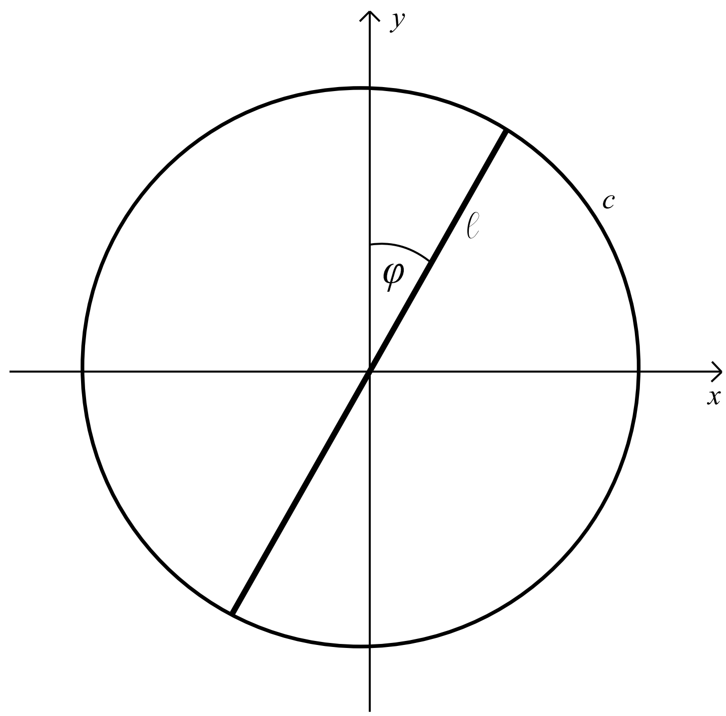

Figure 1: A deflected diagonal.

Now, split the diagonal ℓ into two segments ℓ_1 and ℓ_2. We will choose a direction (either clock-wise, or non clock-wise), and lead n of such directed half-lines (uniformly distant) from either the segment ℓ_1 and ℓ_2 (Figure 2).

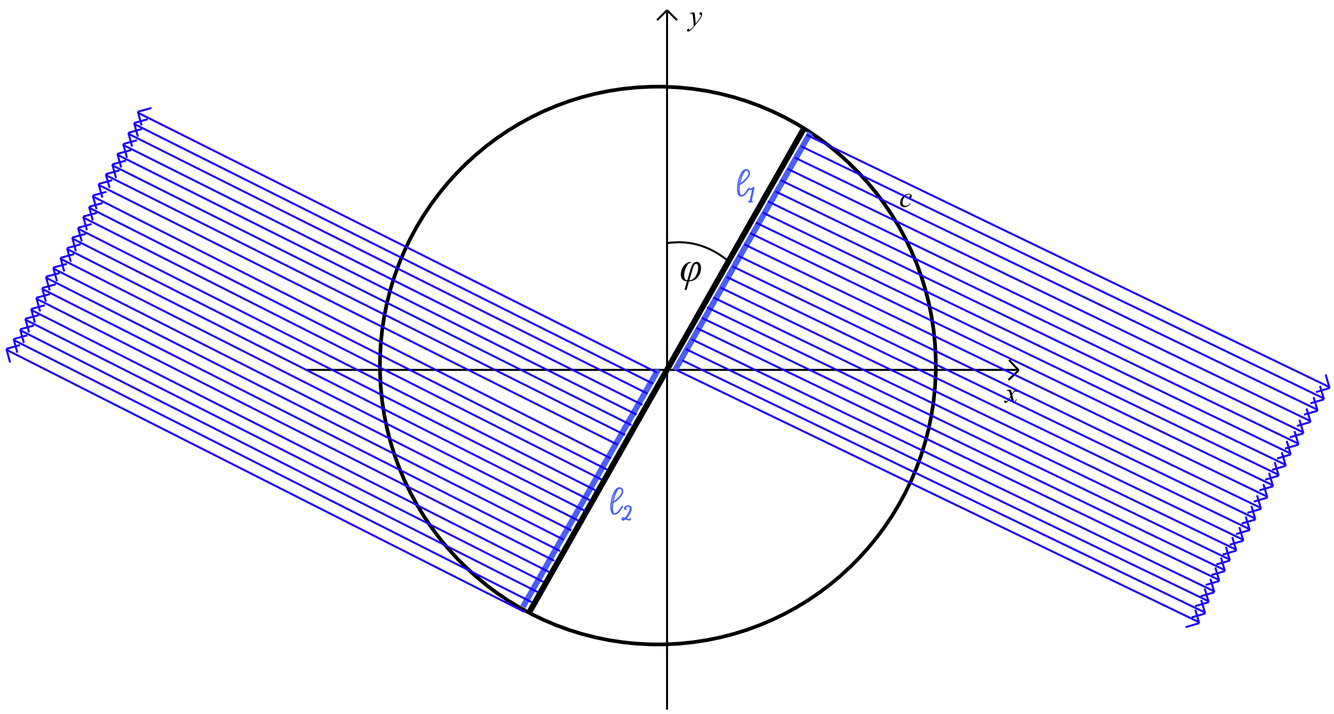

Figure 2: Half-lines led from segments (illustration).

Finally, we add another $k-1$ of diagonals $\ell^1, \ell^2, \ldots, \ell^{k-1}$ such that they will be deflected from initial diagonal ℓ (we can add for indexing $\ell \overset{\text{def}}{=} \ell^0$) by angles $\frac{1}{k}\pi, \frac{2}{k}\pi, \ldots, \frac{k-1}{k}\pi$ consecutively in the chosen direction, and analogically lead n of such directed half-lines (uniformly distant) from either the segment ℓ_1^i and ℓ_2^i for all $i = 0, 1, \ldots, k-1$.

The problem is to find a function $\delta(d)$ of **density of half-line conjunctions led from given k diagonals in distance $d = |Od|$.**

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Solution.
As we are searching for a function of density dependent upon the n of half-lines led, set $n \to \infty$, as this approach provides us the most certain approximation of the function for arbitrary n. This way, geometrically, we transform the problem to searching for an intersection of infinite rectangled areas at the place of led half-lines.

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Lead a half-line h deflected from y-axis by angle φ. Fix a point $d_0 \in h$. We are going to find probability that at point d_0, there will be an intersection of some two of infinite rectangled areas.

Hold: $d_0 \in$ is a point of intersection of two rectangled areas led from segments ℓ_1^r, ℓ_1^s (ℓ_2^r, ℓ_2^s) respectively, denote them S_r, S_s, if and only if $d_0 \in S_r \wedge d_0 \in S_s$. We are finding a range $\mathcal{R}(d)$ of angle deflection from the y-axis for a segment ℓ_i^k such that $d_0 \in S_k$. According to this, the probability function will hold

$$P(d) = \frac{\mathcal{R}(d)}{\pi}.$$

We will consider two cases. First, assume $d \leq r$ (Fig. 3).

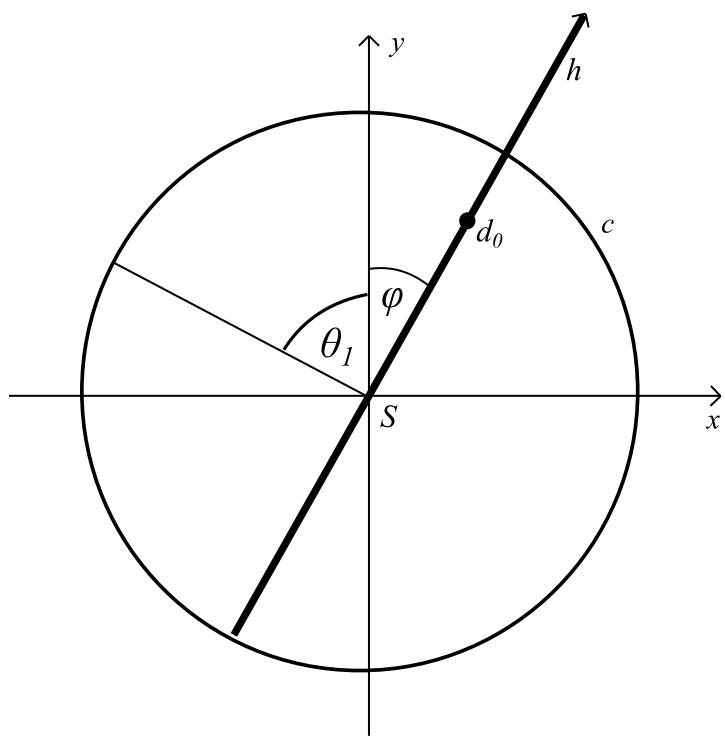

Figure 3: Case of $d \leq r$.

In this case, $\mathcal{R}(d) = \varphi + \theta_1$, and as $\theta_1 = \frac{\pi}{2} - \varphi$, we have $\mathcal{R}(d) = \frac{\pi}{2}$. Then

$$P(d) = \frac{1}{2}.$$

Second, we assume $d > r$ (Fig. 4).

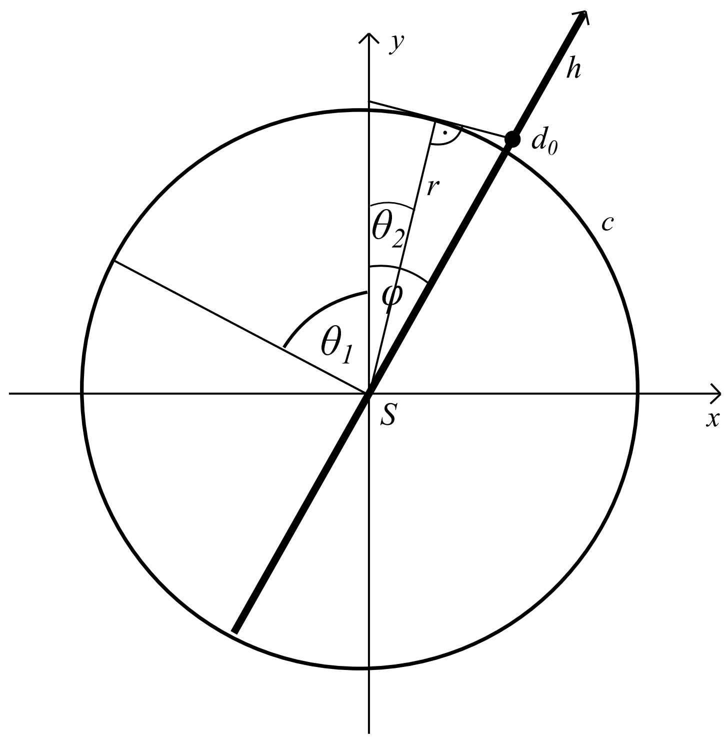

Figure 4: Case of $d > r$.

Hold

$$\frac{d}{\sin(\frac{\pi}{2})} = \frac{r}{\sin\left(\frac{\pi}{2} - \varphi + \theta_2\right)}$$

and as $\theta_2 = \frac{\pi}{2} - \varphi$, this is equal to

$$\frac{r}{\sin\left(\theta_1 + \theta_2\right)}.$$

Of this, we get

$$\mathcal{R}(d) = \theta_1 + \theta_2 = \arcsin\left(\frac{r}{d}\right)$$

such that $\frac{r}{d} \in (0, 1]$. So

$$P(d) = \frac{\arcsin\left(\frac{r}{d}\right)}{\pi}.$$

Probability function $P(d)$ for $d \in \mathbb{R}^+$ is (Figure 5)

$$P(d) = \begin{cases} \frac{1}{2}, & \text{if } d \leq r, \\ \frac{\arcsin\left(\frac{r}{d}\right)}{\pi}, & \text{if } d > r. \end{cases}$$

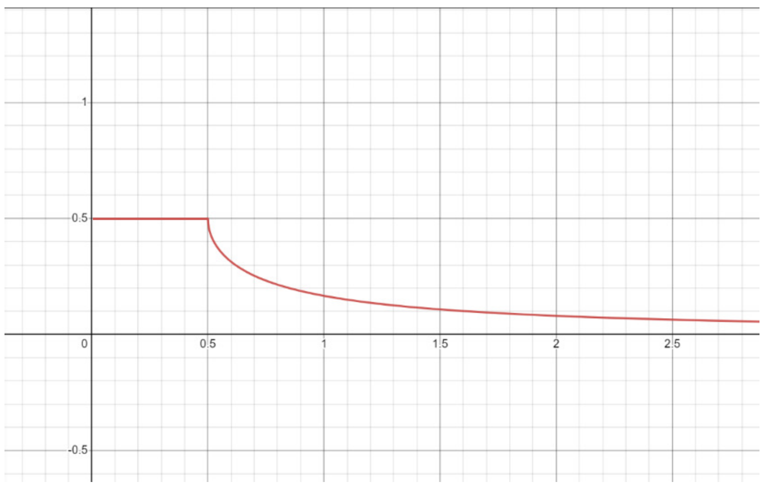

Figure 5: Density probability in respect of d.